Name:____________________________________

CSCI 2824 - Discrete Structures
Test 1

Grade: _____ /100

1. (10 points) For the following pair of propositions show whether P is logically equivalent to Q:

   (a) $P = p \to q$, $Q = \neg p \vee q$

Solution:

p	q	$p \to q$	$\neg p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Since the two columns have the same values, they are logically equivalent.

   (b) $P = p \wedge q$, $Q = \neg q \to p$.

Solution:

p	q	$p \wedge q$	$\neg q \rightarrow p$
T	T	T	T
T	F	F	T
F	F	F	T
F	F	F	F

Since our two highlighted columns differ, these are not logically equivalent.

2. (20 points) For the following parts list each set that has the same cardinality as the given set, from the provided list of sets.

   $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Z} \times \mathbb{Z}$, $\{1, 3\}$, $\{a, 7\}$, $\{1, 2, 4, 5\}$, $\{1, 2, 3, 4, 5\}$, $\{1, \{2, 3\}, \{4\}\}$, $\{7, 132, 19, \{1\}\}$

   (a) E - the set of all even integers.

   Solution: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$.

   (b) $[0, 1]$ - all real numbers between 0 and 1, inclusive.

   Solution: $\mathbb{R}$

   (c) $\{a, b\}$

   Solution: $\{1, 3\}$, $\{a, 7\}$.

   (d) $\{7, 24, 5\}$

Solution: $\{1, \{2, 3\}, \{4\}\}$.

3. (10 points) Let $A(x, y)$ be the propositional function x attended y's office hours and $E(x)$ be the propositional function x is enrolled in a discrete mathematics course. Using these and our quantifiers turn the following statements into symbols that they are equivalent to.

   (a) Brit attended someone's office hours.

   Solution: Either $\exists\, x$ such that $A(\text{Brit}, x)$ or $\exists\, x \exists\, y$ such that $A(x, y)$ were acceptable here.

   (b) Every discrete math student attended someone's office hours.

   Solution: Either $\forall x$ such that $E(x), \exists y$ such that $A(x, y)$ or $\forall x\, \exists y\, E(x) \rightarrow A(x, y)$.

4. (20 points) Prove that $\mathcal{P}(X \cap Y) = \mathcal{P}(X) \cap \mathcal{P}(Y)$ for all sets X and Y.

Solution: To show set equality we show containment both ways.

If $A \in \mathcal{P}(X \cap Y)$ then $A \subseteq X \cap Y$ this means $A \subseteq X$ and $A \subseteq Y$. Since $A \subseteq X$, $A \in \mathcal{P}(X)$, since $A \subseteq Y$, $A \in \mathcal{P}(Y)$. Thus $A \in \mathcal{P}(X) \cap \mathcal{P}(Y)$.

For the reverse direction: if $A \in \mathcal{P}(X) \cap \mathcal{P}(Y)$ then $A \in \mathcal{P}(X)$ so $A \subseteq X$ and we also have $A \in \mathcal{P}(Y)$ so $A \subseteq Y$. Thus $A \subseteq X \cap Y$ meaning that $A \in \mathcal{P}(X \cap Y)$.

$\square$

5. (25 points) Prove that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Solution: Proof by induction. Base case: $n = 1$: $1^2 = \frac{1(2)(3)}{6}$

IH: We assume the identity holds for n and show it holds for $n + 1$ that is we show:

$$1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

By the Inductive hypothesis:

$$\begin{aligned}
1^2 + 2^2 + \cdots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6} \\
&= \frac{(n+1)(n(2n+1) + 6(n+1))}{6} \\
&= \frac{(n+1)(2n^2 + n + 6n + 6)}{6} \\
&= \frac{(n+1)(2n^2 + 7n + 6)}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6}
\end{aligned}$$

This closes the induction.

$\square$

6. (5 points) Define what it means that $f : X \to Y$ is a function.

Solution: $f : X \to Y$ is a function if for every $x \in X$ there is exactly one $y \in Y$ such that $f(x) = y$.

7. (10 points) For functions $f : Y \to Z$ and $g : X \to Y$ prove or disprove the following claim: If $f \circ g$ is onto then g is onto.

Solution: This is false. Choose $X = Z = \{1\}$ and $Y = \{1, 2\}$. Define $g : X \to Y$ by $g(1) = 1$ and define $f : Y \to Z$ by $f(1) = 1$, $f(2) = 1$. Then $f \circ g$ is onto since $f \circ g(1) = f(g(1)) = f(1) = 1$. But g is not onto.