CSCI 2824 - Discrete Structures
Test 1 Review

1. Let the $U = \{1, 2, 3, \ldots, 10\}$ be a universal set and $A = \{1, 4, 7, 10\}$, $B = \{1, 2, 3, 4, 5\}$ and $C = \{2, 4, 6, 8\}$. List the elements of each following set:

   (a) $\overline{A}$

   > **Solution:** $\overline{A} = \{2, 3, 5, 6, 8, 9\}$

   (b) $B \cap C$

   > **Solution:** $B \cap C = \{2, 4\}$

   (c) $\overline{B} \cap (C \backslash A)$

   > **Solution:** First $\overline{B} = \{6, 7, 8, 9, 10\}$ and $C \backslash A = \{2, 6, 8\}$. Thus $\overline{B} \cap (C \backslash A) = \{6, 8\}$

   (d) $(A \cup B) \backslash (C \backslash B)$

   > **Solution:** First $A \cup B = \{1, 2, 3, 4, 5, 7, 10\}$ and $C \backslash B = \{6, 8\}$. Thus $(A \cup B) \backslash (C \backslash B) = \{1, 2, 3, 4, 5, 7, 10\}$

2. Determine the cardinality of the following sets:

   (a) $\emptyset$

   > **Solution:** This is a homework 3 question.

   (b) $\{\emptyset\}$

   > **Solution:** 1

   (c) $\{a, bc, d\}$

   > **Solution:** 3

   (d) $\{a, bc, d, \{a, bc, d\}\}$

   > **Solution:** 4

3. For the following questions determine if $A \subseteq B$ or not.

   (a) $A = \{1, 2\}$, $B = \{3, 2, 1\}$

   > **Solution:** Yes.

   (b) $A = \{1, 2\}$, $B = \{x : x^3 - 6x^2 + 11x = 6\}$

   > **Solution:** Yes, the roots of $x^3 - 6x^2 + 11x = 6$ are $x = 1, 2, 3$.

(c) $A = \{x : x^3 - 2x^2 - x + 2 = 0\}$, $B = \{x^3 - 6x^2 + 11x - 6 = 0\}$

> **Solution:** No, $-1 \in A$ since $-1 - 2 + 1 + 2 = 0$ but $-1 \notin B$ since $-1 - 6 + 11 - 6 = -2$.

(d) $A = \{1, 2, 3, 4\}$, $C = \{5, 6, 7, 8\}$, $B = \{n : n \in A \text{ and } n + m = 8 \text{ for some } m \in C\}$

> **Solution:** We compute $B = \{1, 2, 3\}$. Thus $A \not\subseteq B$.

4. For the following questions represent the following proposition symbolically using the following symbols:

p: There is a hurricane

q: It is raining

(a) There is no hurricane

> **Solution:** $\neg p$.

(b) There is a hurricane, but it is not raining.

> **Solution:** $p \wedge \neg q$.

(c) Either there is a hurricane or it is raining but there is no hurricane.

> **Solution:** $p \vee q \wedge \neg p$

5. Determine the truth value of the following propositions:

(a) If $3 + 5 < 2$ then $1 + 3 = 5$

> **Solution:** True, $3 + 5 > 2$ so the proposition is vacuously true.

(b) $3 + 5 > 2$ if and only if $1 + 3 = 4$.

> **Solution:** True, $3 + 5 > 2$ and $1 + 3 = 4$ are both true statements thus they have the same truth values.

6. Using the following statements write the following propositions in symbols:

p: You run 10 laps daily

q: You are healthy

r: You take multi-vitamins

(a) If you run 10 laps daily, then you will be healthy.

> **Solution:** $p \to q$.

(b) Taking multi-vitamins is sufficient for being healthy

Solution: $r \to q$.

(c) If you are healthy, then you run 10 laps daily or you do not take multi-vitamins.

Solution: $q \to (p \vee \neg r)$.

7. Given that $P(x)$ denotes "x is an accountant" and $Q(x)$ denotes "x owns a Porsche" write each statement symbolically.

(a) All accountants own Porsches

Solution: $\forall x\, P(x) \to Q(x)$

(b) Some accountant owns a Porsche

Solution: $\exists x\, P(x) \wedge Q(x)$.

(c) Someone who owns a Porsche is an accountant.

Solution: $\exists x\, Q(x) \wedge P(x)$.

8. Let $T(x, y)$ stand for the propositional function x is taller than or the same height as y. Write all of the following propositions as words:

(a) $\forall x\, \forall y\, T(x, y)$

Solution: For every pair of people x and y, x is taller or the same height as y.

(b) $\forall x\, \exists y\, T(x, y)$

Solution: For every person x there is a person y such that x is taller than or the same height as y.

(c) $\exists x\, \exists y\, T(x, y)$

Solution: There is a person x and a person y such that x is taller than or the same height as y.

(d) $\exists x\, \forall y\, T(x, y)$

Solution: There is a person x such that for every other person y, x is taller than or the same height as y.

9. Prove the following claim, or provide counter examples:

(a) For all integers m and n, if m and $m + n$ are even then n is even.

Solution: This is true, if m is even then $m = 2k$ for some integer k, if $m+n$ is even then $m + n = 2l$ for some integer l. In particular:

$$\begin{aligned} m + n &= 2l \\ 2k + n &= 2l \\ n &= 2l - 2k \\ n &= 2(l - k) \end{aligned}$$

$l - k$ is an integer so n is even.

□

(b) For sets X, Y, and Z, if $X \subseteq Y$ then $Z \backslash Y \subseteq Z \backslash X$.

Solution: This is true, since X is a "smaller" set we are subtracting less. Proof:
Choose $a \in Z \backslash Y$, this tells us that $a \in Z$ and $a \notin Y$. Since $X \subseteq Y$ we cannot have $a \in X$ either thus $a \in Z \backslash X$.

□

(c) For sets X, Y, and Z, $X \times (Y \backslash Z) = (X \backslash Y) \times (X \backslash Z)$.

Solution: This is not true. Choose $X = \{1, 2, 3, 4\}$, $Y = \{3, 4\}$ and $Z = \{1, 2\}$. Then $(3, 3) \in X \times (Y \backslash Z)$ but $(3, 3) \notin (X \backslash Y) \times (X \backslash Z)$ since $3 \notin X \backslash Y$.

□

(d) $\forall x \in \mathbb{R}$ if x^2 is irrational then x is irrational.

Solution: We prove this via contrapositive. If x is rational then x^2 is rational. If x is rational then $x = \frac{p}{q}$ then $x^2 = \frac{p^2}{q^2}$ which is rational. This proves the original claim.

□

(e) $\sqrt[3]{2}$ is irrational.

Solution: This is proved the same way as that $\sqrt{2}$ is irrational:
Proof by contradiction, suppose that $\sqrt[3]{2}$ is rational then $\sqrt[3]{2} = \frac{p}{q}$ where p and q share no common factors. Cubing both sides we have that $2 = \frac{p^3}{q^3}$ This means that $p^3 = 2q^3$ which gives that p^3 is even. By previous results in class we know this means that p is even as well. Thus $p = 2k$ for some integer k or $p^3 = 8k^3$. Subbing into our above equality: $8k^3 = 2q^3$ meaning that $q^3 = 4k^3$ meaning that q^3 and thus q is even also. But then p and q share a factor of 2 which is against our assumptions. This is our contradiction so the original claim is true.

□

(f) For all $x, y \in \mathbb{R}$ if x is rational and y is irrational then xy is irrational.

Solution: This is not true, $x = 0$ gives that $xy = 0$ which is rational.

□

(g) $(X \backslash Y) \cap (Y \backslash X) = \emptyset$ for all set X and Y.

Solution: This is true. If $a \in X \backslash Y$ then $a \in X$ but not in Y. If $b \in Y \backslash X$ then $b \in Y$ but not in X. To prove this is true we use proof by contradiction. Suppose $a \in (X \backslash Y) \cap (Y \backslash X)$ then $a \in X$ and not in Y (from the first part of our intersection) but the second part tells us that $a \in Y$ and not in X a contradiction. Thus the original claim is true.

□

(h) Let $s_1, s_2, \ldots, s_n$ be any real numbers. We define the average of these numbers as:

$$A = \frac{s_1 + s_2 + \cdots + s_n}{n}$$

Suppose there exists an i and a j such that $s_i \neq s_j$ then there must be some k such that $s_k > A$.

Solution: Proof by contradiction. Suppose that if $s_1, s_2, \ldots, s_n$ are any real numbers and there is some i and some j such that $s_i \neq s_j$ but there is no $s_k > A$. Then either $s_i \neq A$ or $s_j \neq A$, without loss of generality assume that $s_i \neq A$. Then either $s_i > A$ or $s_i < A$. If $s_i > A$ then we are done if not then $s_i < A$ and by our assumption $s_k < A$ for all k meaning:

$$s_1 + s_2 + \cdots + s_n < nA$$

which contradicts A being our average. Thus it must be that some k has $s_k > A$.

□

(i) For all sets A, B, and C, $A \subseteq C$ and $B \subseteq C$ if and only if $A \cup B \subseteq C$.

Solution: This is true. If $A \subseteq C$ and $B \subseteq C$ then for all $a \in A$, $a \in C$ and for all $b \in B$, $b \in C$. Thus for all $d \in A \cup B$ either $d \in A$ (meaning $d \in C$) or $d \in B$ (meaning $d \in C$). So we have $A \cup B \subseteq C$.

For the reverse if $A \cup B \subseteq C$ then this means for all $d \in A \cup B$, $d \in C$. Then for all $a \in A$, $a \in A \cup B$ so $a \in C$ so that $A \subseteq C$. Similarly for all $b \in B$, $b \in A \cup B$ so that $b \in C$ so $B \subseteq C$.

□

(j) For all positive integers n the following holds:

$$\frac{1}{2n} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$$

Solution: Proof by induction. Base case $n = 1$, $\frac{1}{2} \leq \frac{1}{2}$.

IH: For n we have $\frac{1}{2n} \leq \frac{1 \cdot 3 \cdot 5 \cdots 2n+1}{2 \cdot 4 \cdot 6 \cdots (2n)}$.

We show that the inequality holds for $n + 1$:

$$\frac{1}{2n+2} \leq \frac{1}{2n+2} \cdot \frac{2n+1}{2n}$$

This is because $2n + 1 > 2n$ so the fraction is greater than 1

$$= \frac{1}{2n} \cdot \frac{2n+1}{2n+2}$$

Just rearranging the fractions...

$$\leq \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \frac{2n+1}{2n+2}$$

By IH...

$$= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot (2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n) \cdot (2n+2)}$$

This closes the induction.

□

(k) If $X_1, X_2, \ldots, X_n$ and X are sets then:

$$X \cap (X_1 \cup X_2 \cup \cdots \cup X_n) = (X \cap X_1) \cup (X \cap X_2) \cup \cdots \cup (X \cap X_n)$$

Solution: We prove this by induction. Base case $n = 1$ then the equality holds trivially. IH: Assume true for n that is $X \cap (X_1 \cup X_2 \cup \cdots \cup X_n) = (X \cap X_1) \cup (X \cap X_2) \cup \cdots \cup (X \cap X_n)$ then we show true for $n + 1$:

$$X \cap (X_1 \cup X_2 \cup \cdots \cup X_n \cup X_{n+1}) = X \cap ((X_1 \cup X_2 \cup \cdots \cup X_n) \cup X_{n+1})$$

Let $X_1 \cup X_2 \cup \cdots \cup X_n = S$, and remember our induction hypothesis holds true for any sets

$$= X \cap (S \cup X_{n+1})$$

Then our induction hypothesis gives:

$$= (X \cap S) \cup (X \cap X_{n+1})$$

Expanding S:

$$= (X \cap (X_1 \cup X_2 \cup \cdots \cup X_n)) \cup (X \cap X_{n+1})$$

Again our IH:

$$= (X \cap X_1) \cup (X \cap X_2) \cup \cdots \cup (X \cap X_n) \cup (X \cap X_{n+1})$$

This closes the induction.

□

(l) If $X_1, X_2, \ldots, X_n$ and X are sets then:

$$\overline{X_1 \cap X_2 \cap \cdots \cap X_n} = \overline{X_1} \cup \overline{X_2} \cup \cdots \cup \overline{X_n}$$

Solution: Proof by induction. Base case, $n = 1$ this is trivially true.
IH: Assume true for n that is assume $\overline{X_1 \cap X_2 \cap \cdots \cap X_n} = \overline{X_1} \cup \overline{X_2} \cup \cdots \cup \overline{X_n}$ then we show this holds for $n + 1$:

$$\overline{X_1 \cap X_2 \cap \cdots \cap X_n \cap X_{n+1}} = \overline{A \cap X_{n+1}}$$

Where we let $A = X_1 \cap X_2 \cap \cdots \cap X_n$, now using our induction hypothesis (which holds for any sets)

$$= \overline{A} \cup \overline{X_{n+1}}$$

Expanding A:

$$= \overline{X_1 \cap X_2 \cap \cdots \cap X_n} \cup \overline{X_{n+1}}$$

Using our induction hypothesis again:

$$= \overline{X_1} \cup \overline{X_2} \cup \cdots \cup \overline{X_n} \cup \overline{X_{n+1}}$$

This closes the induction.

□

10. Let f and g be functions from the positive integers to the positive integers defined by

$$f(n) = 2n + 1, \qquad g(n) = 3n - 1$$

Find the compositions $f \circ f$, $g \circ g$, $f \circ g$ and $g \circ f$.

Solution:

$$\begin{aligned} f \circ f(n) &= f(f(n)) \\ &= f(2n + 1) \\ &= 2(2n + 1) + 1 \\ &= 4n + 3 \end{aligned}$$

$$\begin{aligned} g \circ g(n) &= g(g(n)) \\ &= g(3n - 1) \\ &= 3(3n - 1) - 1 \\ &= 9n - 4 \end{aligned}$$

$$
\begin{aligned}
f \circ g(n) &= f(g(n)) \\
&= f(3n-1) \\
&= 2(3n-1)+1 \\
&= 6n-1
\end{aligned}
$$

$$
\begin{aligned}
g \circ f(n) &= g(f(n)) \\
&= g(2n+1) \\
&= 3(2n+1)-1 \\
&= 6n+2
\end{aligned}
$$

11. Each of the following functions is one-to-one and thus is a bijective function from its domain to its image. Find the inverse of the given functions.

(a) $f(x) = 4x + 2$, $x \in \mathbb{R}$.

Solution: $f^{-1}(x) = \frac{x-2}{4}$

(b) $f(x) = 3\log_2(x)$, $x \in \mathbb{R}^{>0}$.

Solution: $f^{-1}(x) = 2^{\frac{x}{3}}$

(c) $f(x) = 6 + 2^{7x-1}$, $x \in \mathbb{R}$.

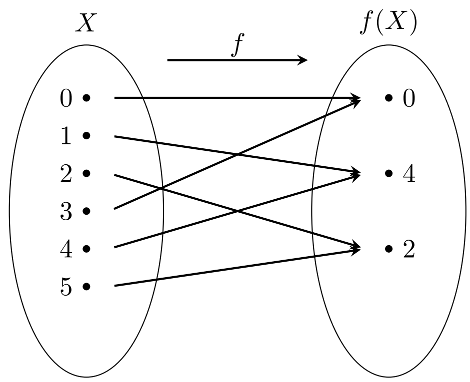
Solution: $f^{-1}(x) = \frac{\log_2(x-6)+1}{7}$

12. Let f be the function from $X = \{0,1,2,3,4,5\}$ to X defined by

$$f(x) = 4x \mod 6$$

Write f as a set of ordered pairs and draw the arrow diagram for f. Is f injective? Surjective?

Solution: $f = \{(0,0),(1,4),(2,2),(3,0),(4,4),(5,2)\}$
The arrow diagram for this is:



This function is not injective, both 0 and 3 map to 0. This is not surjective since 3 is not hit by any input.

13. For the following questions let g be a function from X to Y and let f be a function from Y to Z. For each of the following if the statement is true prove it otherwise proved a counterexample.

(a) If g is onto then $f \circ g$ is onto.

Solution: This is not true, let $X = Y = Z = \{1, 2, 3\}$ define $f(x) = 1$ for all $x \in Y$ and let $g(x) = x$ for all $x \in X$. Then g is onto, but f is not. Then $f \circ g$ is not onto because there is no input which will map to 2 (all inputs eventually go to Rome. . . I mean 1).

(b) If $f \circ g$ is injective then f is injective.

Solution: This is not true. Consider $Y = Z = \{1, 2, 3\}$ and $X = \{1, 2\}$ and let $f(1) = 1, f(2) = 1$, and $f(3) = 2$. f is clearly not injective. Define $g(1) = 2$ and $g(2) = 3$. g clearly is injective. Then we have that no two inputs to $f \circ g$ map to the same output, so that $f \circ g$ is injective, but f is not.

(c) If f is one-to-one then $f \circ g$ is one-to-one.

Solution: Again not true. Let $X = Y = Z = \{1, 2, 3\}$ and define $f(x) = x$ for all $x \in Y$ then f is one-to-one. Define $g(1) = 1, g(2) = 1$, and $g(3) = 2$. $f \circ g(1) = 1$ and $f \circ g(2) = 1$ meaning that $f \circ g$ is not one-to-one.