Summations

We often discuss adding a sequence together:

$$a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

This needs a notation, we use $\sum_{j=m}^{n} a_j$

j is our index of summation.

Usual arithmetic operations apply: $\sum_{j=1}^{n} a \cdot x_j + b \cdot y_j = a \sum_{j=1}^{n} x_j + b \sum_{j=1}^{n} y_j$

Ex: what is the value $\sum_{j=1}^{5} j^2$?

$$= 1 + 4 + 9 + 16 + 25 = 55.$$

Sometimes we need/want our summation to begin/end at specific terms. i.e. from above we want it to begin at 0. Then we need to index shift

$$\sum_{j=0}^{4} (j+1)^2 = \sum_{k=1}^{5} k^2$$

<u>Theorem</u>: If $a, r \in \mathbb{R}$ & $r \neq 0$ then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

<u>Pf</u>: Let $S_n = \sum_{j=0}^{n} ar^j \Rightarrow rS_n = \sum_{j=0}^{n} ar^{j+1}$

$$rS_n = \sum_{k=1}^{n+1} ar^k$$

$$= \sum_{k=0}^{n} ar^k + (ar^{k+1} - a) \quad \left[\text{take away } k+1 \text{ term add 0 term}\right]$$

$$= S_n + ar^{n+1} - a \quad \left[S_n = \sum_{j=0}^{n} ar^k\right]$$

So $rS_n = S_n + ar^{n+1} - a$

$$\Rightarrow S_n(r-1) = ar^{n+1} - a$$

$$S_n = \frac{ar^n - a}{r-1} \quad (\text{only if } r \neq 1 \text{ otherwise can't divide})$$

If $r=1$, $S_n = \sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} a \cdot 1 = (n+1)a$. $\square$

We often need to sum specific terms from sets, this is written

$$\sum_{s \in S} f(s)$$

Ex: Determine the sum of squares of odd numbers between 1 & 9

$\Rightarrow S = \{1, 3, 5, 7, 9\}$

$$\sum_{s \in S} s^2 = 1^2 + 9 + 25 + 49 + 81 = ?$$

Ex. Given $\displaystyle\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$ Determine $\displaystyle\sum_{k=50}^{100} k^2$.

Could index shift, but then lose k^2.

Instead $\displaystyle\sum_{k=1}^{100} k^2 = \sum_{k=1}^{49} k^2 + \sum_{k=50}^{100} k^2 \Rightarrow \sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2$

$$= \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6} = 338,350 - 40,425 = 297,925.$$

We could talk about infinite sums as well, but this gets very complicated.

See Calculus 2.