Sets

Now were gonna spend some time building all the fundamental structures that we'll need.

Def: A set is an unordered collection of objects, called elements.
A set contains elements. we write $a \in A$ to denote a is an element of set A, or $a \notin A$ for a is not in A.

Sets are usually defined by capital letters.

To describe ~~the~~ the elements of a set we often list them inside curly braces

ex: $A = \{1, 2, 3, 4\}$

Often when working with sets a given set collects similar objects
this is not necessary.

ex: $B = \{$ John, "Hello", 5, 10^{70}, Colorado $\}$ is a perfectly valid set.

For brevity we occasionally use ... in sets:

ex $C = \{1, 3, 5, \ldots, 13\}$

The more common method of describing sets is set builder notation

Ex $C = \{x : 1 \leq x \leq 13$ and x is odd$\}$

This describes the set instead of listing all elements.

Important sets: Some groups of numbers are used repeatedly & thus get special notation: $\mathbb{N} = \{0, 1, 2, \dots\}$ called the natural numbers

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ set of Integers

$\mathbb{Z}^+$ positive integers $0 \notin \mathbb{Z}^+$

$\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$ The rationals

$\mathbb{R}$ real numbers

$\mathbb{R}^+$ positive real numbers

$\mathbb{C}$ complex numbers $= \{a+bi : a, b \in \mathbb{R}\}$

Intervals are also common: $[a,b] = \{x : x \in \mathbb{R},\ a \le x \le b\}$

$[a,b) = \{x : x \in \mathbb{R},\ a \le x < b\}$

$(a,b] = \{x : x \in \mathbb{R}\ \ a < x \le b\}$

$(a,b) = \{x : x \in \mathbb{R}\ \ a < x < b\}$

$\mathbb{N}, \mathbb{R}, \mathbb{Z}$ etc have infinitely many elements.

Ex $\{\mathbb{N}, \mathbb{R}, \mathbb{Z}\}$ has 3 elements.

Def: Two sets are said to be equal if they have the same elements

$A = B$ iff $\forall x\ (x \in A \leftrightarrow x \in B)$.

Ex $\{1, 3, 5\}$, $\{3, 1, 5\}$ are equal sets.

Note, we generally do not allow elements to be repeated in sets

$\{1, 1, 3, 5\} = \{1, 3, 5\}$,

There is One Set which has no elements: The empty set (or null set)

$\{\}$, $\emptyset$.

Note $\{\emptyset\}$ has one element, it contains the empty set.

Def A set containing one element is often called ~~a~~ Singleton

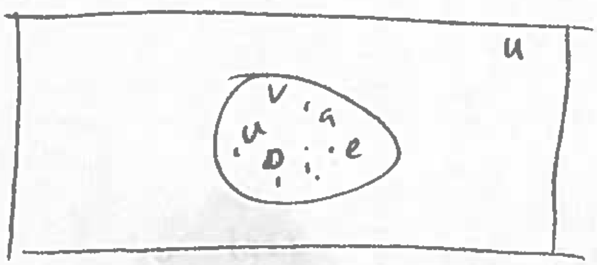
Ex: $\{a\}$, $\{5\}$, $\{\emptyset\}$ are all single ton sets.

Sets can be described pictorially:

U - our universe (all possible elements)

[in this case U = all letters]

V = our set with elements.



Subsets: We often want to be able to describe when one set is part of another.

Def: The set A is a subset of the set B, iff every element of A is also an element of B. We use $A \subseteq B$ to write A is a subset of B.

$$A \subseteq B \text{ iff } \forall x\,(x \in A \to x \in B)$$

To show $A \subseteq B$ we must show for every $x \in A \Rightarrow x \in B$

To show $A \not\subseteq B$ we must find one $x \in A \Rightarrow x \notin B$.

Ex: $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

Ex: $\mathbb{C} \not\subseteq \mathbb{R}$

Ex: $\{1,3\} \subseteq \{1,3,5\}$

Ex: $\{1,3\} \not\subseteq \{1,5\}$

<u>Ex</u>: $\{1,3\} \overset{\in}{\not\subseteq} \{\{1,3\},1,5\}$

<u>Ex</u>: $\{1,3\} \overset{\subseteq}{\notin} \{1,3,5\}$

Sometimes we wish to point out a subset is strictly smaller than another.

$\{1,3\} \subset \{1,3,5\} \Rightarrow \{1,3\} \subseteq \{1,3,5\}$ & $\{1,3\} \neq \{1,3,5\}$

or $\subsetneq$

<u>Ex</u>: $X = \{x : x^2 + x - 2 = 0\} \in \mathbb{Z}$

$x^2 + x - 2 = 0$

$(x-1)(x+2) = 0$

$x = 1, -2$

<u>Theorem</u>: For every set S, (i) $\emptyset \subseteq S$ (ii) $S \subseteq S$.

<u>Pf</u>: (i) To show $\emptyset \subseteq S$ we must show $\forall x \ (x \in \emptyset \rightarrow x \in S)$

but $x \in \emptyset$ is always false since $\emptyset$ is empty $\Rightarrow$ $\forall x \ (x \in \emptyset \rightarrow x \in S)$

is vacuously true. Thus $\emptyset \subseteq S$.

(ii). $\forall x \ (x \in S \rightarrow x \in S)$: If $x \in S$ then $x \in S$.

$\square$

"Shortcut" on set equality. Instead of proving logical statements

we simply show $A \subseteq B$ & $B \subseteq A \Rightarrow A = B$.

We often wish to discuss the size of a set. This is called the cardinality of the set. This is often denoted as $|S|$.

Ex: Let S = set of letters of English alphabet $\Rightarrow$ $|S| = 26$.

Ex: $|\emptyset| = 0$, $|\mathbb{R}| = \infty$ (we'll discuss infinity on Wednesday).

Def: Given a set S the power set of S is the set of all subsets of S, Denoted $\mathcal{P}(S)$.

This is useful for looking at all combinations of elements of a set.

Ex: $\mathcal{P}(\{1,2,3\}) = ?$ $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

1. $\emptyset \subseteq \{1,2,3\}$
2. $\{1\} \subseteq \{1,2,3\}$
3. $\{2\}$
4. $\{3\}$
5. $\{1,2\}$
6. $\{1,3\}$,
7. $\{2,3\}$
8. $\{1,2,3\}$

Ex: $\mathcal{P}(\emptyset) = ?$ $\mathcal{P}(\{\emptyset\}) = ?$

$\{\emptyset\}$. $\{\emptyset, \{\emptyset\}\}$

The power set of a finite set with n elements has 2^n elements

_ _ _ _ ... _

for each of the n elements a subset has a binary choice, is that element in or out?

So All subsets must have all possible options.