Operations on Sets: If sets were just objects they wouldn't be useful, we need to be able to do things with them.

Our first operation is Cartesian products.

Def: The ordered n-tuple $(a_1, a_2, a_3, \ldots, a_n)$ is an ordered collection.

Two n-tuples equal iff they agree on each component.

Note order is important here! $(1,2) \neq (2,1)$.

Def: Let A, B be sets. The Cartesian product of A & B, denoted as $A \times B$ is the set of all ordered pairs (a,b) where $a \in A$, $b \in B$

$$A \times B = \{(a,b) : a \in A, b \in B\}$$

Ex: If $X = \{1,2,3\}$ & $Y = \{a,b\}$ then

$$X \times Y = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}.$$

Note: $Y \times X \neq X \times Y$

$$\{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$$

The elements are not the same. $(a,1) \neq (1,a)$.

No reason we need to stick with two sets:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) : a_i \in A_i\}$$

Ex: If $X = \{1, 2\}$, $Y = \{a, b\}$, $Z = \{\alpha, \beta\}$

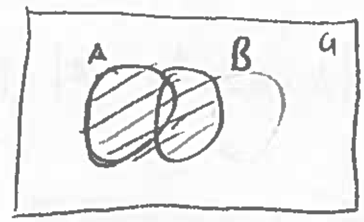
Then $X \times Y \times Z = \{(1, a, \alpha), (1, a, \beta), \dots, (2, b, \beta)\}$.

Note $|A_1 \times A_2 \times A_3 \times \dots \times A_n| = |A_1| \cdot |A_2| \cdot |A_3| \cdots |A_n|$.

Def: Let A & B be sets. The union of A & B is the set $A \cup B$.

This contains elements in A, B, or both.

$A \cup B = \{x : x \in A \text{ or } x \in B\}$.



[///] $= A \cup B$

Ex: $A = \{1, 3, 5\}$ $B = \{2, 4, 6\}$

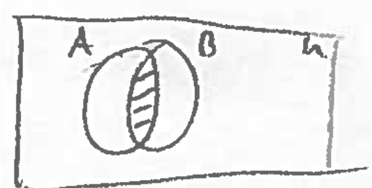
$A \cup B = \{1, 2, 3, 4, 5, 6\}$.

Ex: $A = \{1, 2\}$, $B = \{2, 3\}$

$A \cup B = \{1, 2, 3\}$ (no repetition!)

Def: Let A & B be sets. The intersection of A & B is the set $A \cap B$ which contains elements in both A & B.

$A \cap B = \{x : x \in A \text{ \& } x \in B\}$



[//] $= A \cap B$.

Ex: $A = \{1, 3, 5\}$ $B = \{2, 4, 6\}$

$A \cap B = \emptyset$

We call sets disjoint if their intersections are empty.

Ex: $A = \{1, 2\}$ $B = \{2, 3\}$

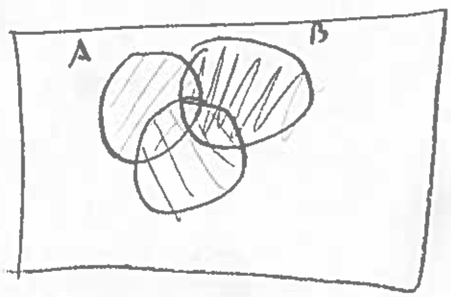
$A \cap B = \{2\}$.

We can count the number of elements in $A \cup B$:

$$|A \cup B| = |A| + |B| \underbrace{- |A \cap B|}_{\text{wait for class to see problem.}}$$

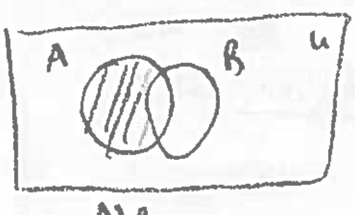
This is generalizable, but gets messy:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



<u>Def</u>: Let A & B be sets. The difference of A & B is denoted $A - B$ or $A \setminus B$. $A \setminus B$ is the set containing elements from A but not in B.

$$A \setminus B = \{ x ; x \in A \wedge x \notin B \}$$



$A \setminus B$

<u>Ex</u>: $\{1,3,5\} \setminus \{4,5,6\} = \{1,3\}$

$\{4,5,6\} \setminus \{1,3,5\} = \{4,6\}$

Def: Let U be a universal set. The complement of a set A, denoted $\bar{A}$ or $\tilde{A}$ is the set $U \setminus A$.

$$\bar{A} = \{x : x \in U \ \& \ x \notin A\}$$

Ex: If $U = \{1,2,3,4,5\}$ & $A = \{1,3,5\}$ then $\bar{A} = \{2,4\}$

if $U = \{1,3,5,7,9\}$ then $\bar{A} = \{7,9\}$

Note $A \subseteq U$ always. Can't have a set not have elements from universal set.

Ex If $\mathbb{Z} = U$ then $\bar{\mathbb{Z}} = \mathbb{N}$.

Ex Show $A \setminus B = A \cap \bar{B}$.

Pf: To show set equality we show the two are subsets of each other.

($\subseteq$) If $x \in A \setminus B$ then $x \in A$ & $x \notin B \Rightarrow x \in \bar{B} \Rightarrow x \in A \cap \bar{B}$

($\supseteq$) If $x \in A \cap \bar{B}$ then $x \in A$ & $x \in \bar{B} \Rightarrow x \notin B$ So $x \in A \setminus B$. $\square$

There is a big list of identities in the discrete Math book

don't need to memorize, but you should be able to prove the equality.

Set proofs can also be done via set builder notation:

Ex show $\overline{A \cap B} = \bar{A} \cup \bar{B}$

pf: $\overline{A \cap B} = \{x : x \notin A \cap B\}$

$= \{x : \neg(x \in A \cap B)\}$

$= \{x : \neg(x \in A \wedge x \in B)\}$

$= \{x : x \notin A \vee x \notin B\}$

$= \{x : x \in \bar{A} \vee x \in \bar{B}\}$

$= \{x : x \in \bar{A} \cup \bar{B}\}$

$= \bar{A} \cup \bar{B}$.

$\square$

Personally, seems not as clean to me, but whatever works for you.

Sometimes there are more than two sets we want to combine.

If we're using the same operation, order doesn't matter; & we have special notation

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i \qquad A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

Ex: Let $A_i = \{i, i+1, i+2, \dots\}$ for all $i \geq 1$.

$A_7 = \{7, 8, 9, \dots\}$.

$$\bigcup_{i=1}^{n} A_i = \{1, 2, 3, \dots\} \qquad \bigcap_{i=1}^{n} A_i = \{n, n+1, n+2, \dots\}$$

Can do this infinitely,

$$\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}^+ \qquad \bigcap_{i=1}^{\infty} A_i = \emptyset$$