Sequences

Def: A sequence is an ordered list i.e. 1, 2, 3, 5, 8 is a sequence with 5 terms

1, 3, 9, 27, 81, ..., 3^n, ... is an infinite sequence.

We often write a sequence as $\{a_n\}$ or $\{a_n\}_{n=1}^{k}$ for a finite sequence.

a_n is the nth element of the sequence. Despite the curly braces $\{a_n\}$ is **not** a set, it is ordered.

Ex: Consider the sequence $u_n = n^2 - 1 \quad n \geq 0$.

$u_0 = -1 \quad u_1 = 0 \quad u_2 = 3$ etc.

Def: A geometric progression is a sequence of the form

$a, ar, ar^2, ar^3, \ldots, ar^n, \ldots$

where a is the initial term & r is the common ratio

Ex: $c_n = 3 \cdot 5^n$ the initial term is 3 & the common ratio is 5.

$c_0 = 3, \ c_1 = 15, \ c_2 = 75, \ c_3 = 375$

notice $c_{i+1}/c_i = 5$ always.

Recurrence relations: A recurrence relation is a sequence defined recursively.

Ex: $a_n = a_{n-1} + 3$ for $n \geq 1$.

$a_0 = ?$ Need a base case $a_0 = 2 \Rightarrow a_1 = 5, \ a_2 = 8, \ldots$

Note: You need as many initial conditions as you have previously referenced terms.

Ex $a_n = a_{n-2} + 5$ needs 2 initial conditions

Ex Fibonacci Sequence

$f_n = f_{n-1} + f_{n-2}$ needs how many ICs? 2

$f_1 = f_2 = 1$

$f_3 = f_2 + f_1 = 2$ what about $f_{100} = ?$

$f_4 = f_3 + f_2 = 3$ need f_{99} & f_{98}

$f_5 = f_4 + f_3 = 5$

$f_6 = f_5 + f_4 = 8$

Computer ex In csci2824 folder. fib1.py vs fib2.py.

Ex: A person invests \$1000 at 12% interest compounded anually. If A_n represents the amount of money at the end of n years find a recurrence relation & initial condition that defines $\{A_n\}$. Can you find a closed form?

Sol: At $n-1$ years we'll have A_{n-1} dollars. To get A_n we need to add A_{n-1} plus interest. So

$$A_n = A_{n-1} + (0.12)(A_{n-1}) = 1.12\,A_{n-1} \quad \forall n \geq 1.$$

& $A_0 = 1000$.

To find a closed form we need to find a pattern w/o recurrence relations.

To see how lets start trying to find a pattern:

$$A_1 = (1.12)(1000) = 1120.0$$
$$A_2 = (1.12)(1120.0) = 1254.4 = (1.12)(1.12)(1000)$$
$$A_3 = (1.12)(1254.4) = 1404.93 = (1.12)(1.12)(1.12)(1000)$$

So $A_n = (1.12)^n (1000)$.

Solving some recurrence relations (as above) are easy. Usually when only one previous term shows up. You need a specific method if two terms show up.

Def: A linear homogeneous recurrence relation of order k w/ constant coeffs is of the form:

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k}$$

Order k relations require k I.C.s.

order k = reference k previous terms.

linear = no terms multiplied together. $A_n = A_{n-1} A_{n-2}$

homogeneous = only relation terms. $A_n = A_{n-1} + 5$ or $+ 5\sin(n)$

Constant coeffs = no fcn coeffs $A_n = n A_n$

we're going to learn how to solve order 2 linear homogeneous recurrence relations w/ constant coeffs.

Theorem: Let $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ be a second order homogeneous linear recurrence relation w/ constant coeffts.

If S & T are solutions then $U = bS + dT$ is also a solution.

(ii) If r is a root of

$$t^2 - c_1 t - c_2 = 0 \qquad (*)$$

then r^n is a solution to the relation.

If r_1 & r_2 are roots of $(*)$ ~~then~~ $r_1 \neq r_2$ then

$$a_n = b r_1^n + d r_2^n \text{ is a solution.}$$

<u>Pf</u>: If S & T are solutions

$$S_n = c_1 S_{n-1} + c_2 S_{n-2} \quad \& \quad T_n = c_1 T_{n-1} + c_2 T_{n-2}$$

then multiply S by b & T by d & add

$$U_n = b S_n + d T_n = c_1 (b S_{n-1} + d T_{n-1}) + c_2 (b S_{n-2} + d T_{n-2})$$
$$= c_1 U_{n-1} + c_2 U_{n-2}$$

Thus U solves the same equation.

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(ii) Since r solves the polynomial

$$r^2 = c_1 r + c_2 \qquad \text{multiplying by } r^{n-2}:$$

$$r^{n-2} \cdot r^2 = r^{n-2}(c_1 r + c_2)$$

$$r^n = c_1 r^{n-1} + c_2 r^{n-2}$$

Thus $\{r^n\}$ solves the above sequence.

(iii) follows from (i) & (ii).

$\square$

Ex: Find a closed form for the Fibonacci Sequence

$f_n = f_{n-1} + f_{n-2}$ $\qquad f_1 = f_2 = 1$

$t^2 - t - 1 = 0 \Rightarrow t = \dfrac{1 \pm \sqrt{1-4(1)(-1)}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

so $f_n = b\left(\dfrac{1+\sqrt{5}}{2}\right)^n + d\left(\dfrac{1-\sqrt{5}}{2}\right)^n$ using I.CS

$$\left.\begin{array}{ll} n=1 & 1 = b\left(\dfrac{1+\sqrt{5}}{2}\right) + d\left(\dfrac{1-\sqrt{5}}{2}\right) \\ n=2 & 1 = b\left(\dfrac{1+\sqrt{5}}{2}\right)^2 + d\left(\dfrac{1-\sqrt{5}}{2}\right) \end{array}\right\} \Rightarrow \text{math} \Rightarrow b = \frac{1}{\sqrt{5}} \quad d = \frac{-1}{\sqrt{5}}$$

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Ex: Solve $a_n = 7a_{n-1} - 10a_{n-2}$; $a_0 = 5$, $a_1 = 16$

$t^2 - 7t + 10 = 0 \qquad (t-5)(t-2) = 0$

$t = 2, 5$

$a_n = b \cdot 2^n + d \cdot 5^n$

$n=0 \quad 5 = b + d \qquad n=1 \quad 16 = b \cdot 2 + 5 \cdot d$

$b = 5 - d \Rightarrow 16 = (5-d)2 + 5d$

$= 3d + 10$

$6 = 3d \Rightarrow d = 2 \Rightarrow b = 3 \qquad a_2 = 7 \cdot 16 - 10 \cdot 5 = 62$

$a_n = 3 \cdot 2^n + 2 \cdot 5^n$ $\qquad$ Check $a_2 = 3 \cdot 4 + 2 \cdot 25 = 62$ ✓

will always work

$a_3 = 7 \cdot 62 - 10 \cdot 16 = 274$

Check $a_0 = 3 + 2 = 5$ ✓ $\quad a_1 = 3 \cdot 2 + 2 \cdot 5 = 16$ ✓ $\qquad a_3 = 3 \cdot 8 + 2 \cdot 125 = 274$ ✓