Relations

We'll briefly talk about relations today. A relation is a generalization of a function. Remember a function assigned every element of the domain to exactly one element of the Co-domain. A function was a subset of Domain $\times$ Codomain.

<u>Def</u>: Let A & B be sets. A <u>binary relation</u> from A to B is a subset of $A \times B$.

That is <u>any</u> subset of $A \times B$. If R is our relation $R \subseteq A \times B$ we say a is related to b if $(a,b) \in R$. We also write aRb. If a is not related to b we might write $(a,b) \notin R$ or $a \not{R} b$.

<u>Ex</u>: If we let $X = \{$ Bill, Mary, Beth, Dave $\}$

$Y = \{$ CompSci, Math, Art, History $\}$

We can write a relation describing people's interests:

$R = \{$ (Bill, Math), (Mary, CS), (Bill, Art), (Beth, History), (Beth, CS), (Mary, Math) $\}$

So we might write Beth R CS.

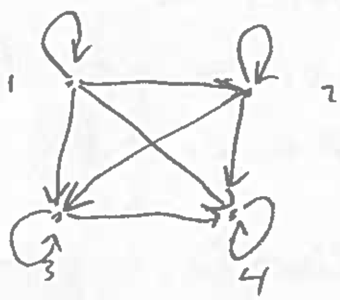
Other thing of note: yes a relation NOT a function! Dave is not related to anything & each of the others are related to multiple things. $\Rightarrow$ Not okay for function, but fine for relation.

Ex: Let R be the relation on $X = \{1,2,3,4\}$ defined by $(x,y) \in R$ if $x \leq y$ for $x, y \in X$. Then

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$$

Note often relations act on only one set. So we say it is a relation on X.

Often it's helpful to draw Digraphs - graphs with direction to explain Relations:



arrow $a \to b \Rightarrow (a,b) \in R$.

Ex: If $A = \{1,2,3,4\}$ $R = \{(a,b) : a \mid b\}$ what is R?

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$$

Def: A relation R on A is said to be *reflexive* iff $(a,a) \in R$ $\forall a \in A$.

e.g. i.e. every node has a self loop.

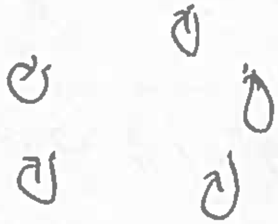
Def: A relation R on A is called *symmetric* if whenever $(a,b) \in R$ then $(b,a) \in R$ $\forall a,b \in A$. A relation R on A such that for all $a,b \in A$ if $(a,b) \in R$ & $(b,a) \in R$ then $a=b$ is called *anti symmetric*.

Symmetric means if a is related to b then b is also related to a (all arrows go both ways).

Anti symmetric means the only arrows that go both ways are loops i.e. if a is related to b & b is related to a then $a=b$

<u>Ex</u>: $A=\{1,2,3,4,5\}$ $\quad R=\{(a,b)\in A\times A : a=b\}$

R is symmetric & anti symmetric



<u>Ex</u>: $R=\{(a,b)\in A\times A : a+b\le 5\}$

$R=\{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3)\dots\}$

if $(a,b)\in R$ then $a+b\le 3 \Rightarrow b+a\le 3 \Rightarrow (b,a)\in R$

So R is symmetric.

R is not anti-symmetric $(1,2)\in R \Rightarrow (2,1)$ but $1\neq 2$.

Not reflexive $(4,4)\notin R$.

<u>Def</u>: A relation R on A is called <u>transitive</u> if whenever $(a,b)\in R$ and $(b,c)\in R$ then $(a,c)\in R$ $\forall a,b,c\in A$.

$R=\{(a,b)\in A\times A : a\le b\}$ is transitive.

if $(a,b)\in R$ & (b,c) then $a\le b\le c \Rightarrow a\le c \Rightarrow (a,c)\in R$.