Quantifiers pt2:

Quantifiers can restrict the domain: $\forall x < 0 \; (x^2 > 0)$

for all $x < 0$ s.t. $x^2 > 0$

Note: This is the same as $\forall x \, (x < 0 \rightarrow x^2 > 0)$

Ex: $\exists z > 0 \; (z^2 = 2)$

There exists a $z > 0$ s.t. $z^2 = 2$

This is the same as $\exists z \, (z > 0 \wedge z^2 = 2)$.

Equivalence of Quantified statements

Ex: Show $\exists x \, (P(x) \vee Q(x))$ is logically equivalent to $\exists x P(x) \vee \exists x \, Q(x)$.

Here we need to show the first is true whenever the second is true & vice versa.

Pf: $(\Rightarrow)$ If $\exists x \, (P(x) \vee Q(x))$ is true then $\exists x$ s.t. either $P(x)$ or $Q(x)$.

$\Rightarrow \exists x \, P(x)$ or $\exists x \, Q(x) \rightarrow \exists x \, P(x) \vee \exists x \, Q(x)$.

$(\Leftarrow)$ If $\exists x \, P(x) \vee \exists x \, Q(x)$ is true then either $\exists x \, P(x) \Rightarrow$

$\exists x \, (P(x) \vee Q(x))$ or $\exists x \, Q(x) \Rightarrow \exists x \, (P(x) \vee Q(x))$

or both.

$\square$.

Ex: Determine whether $\forall x\,(P(x) \leftrightarrow Q(x))$ is logically equivalent to $\forall x\, P(x) \leftrightarrow \forall x\, Q(x)$.

Note: in the second we can chose different x's.

e.g. $P(x)$ = x is even $Q(x)$ = x is a multiple of 2.

Then $\forall x\,(P(x) \leftrightarrow Q(x))$ is true, for any x $P(x)$ & $Q(x)$ have the same value.

The second says for any x (say $x=4$) $P(x)$ is the same as any $Q(x)$ (say $x=5$) which is false.

Nested Quantifiers

Now we make things more complicated! Nesting our quantifiers.

e.g. $\forall x \exists y\,(x+y=0)$

This is read outside in: For every x there is some y s.t. $x+y=0$.

Since y is second it can depend on x. x is first it must be independent of everything.

Think of this as a challenge: For every x you give me, I can find a y s.t. $x+y=0$. The answer is obviously $y=-x$.

Order matters: Let $Q(x,y) = x \leq y$. on all integers.

Give the truth values for $\forall x \exists y\, Q(x,y)$ & $\exists x \forall y\, Q(x,y)$.

a) $\forall x \exists y\, Q(x,y)$. is True. For any x we can choose $y = x + \cancel{1}$.

b) $\exists x \forall y\, Q(x,y)$ is False. For any x we can find a y s.t. $\neg Q(x,y)$.

To disprove this I needed to prove the negation of it.

Negating Quantified Expressions:

Consider "Every student in your class has taken a CS course"

We can write this as ... ?

$\forall x\ P(x)$ x is students in your class. $P(x)$ is x has taken a CS course.

What is the negation in English? "There is a student in your class who has not taken a CS course" or $\exists x\ \neg P(x)$.

Claim: $\neg(\forall x\ P(x)) \equiv \exists x\ \neg P(x)$.

Pf: $(\Rightarrow)$ if $\neg(\forall x\ P(x))$ is true $\Rightarrow$ $\forall x\ P(x)$ is false which means $\exists x\ \neg P(x)$

$(\Leftarrow)$ If $\exists x\ \neg P(x)$ is true $\Rightarrow$ $\forall x\ P(x)$ is false $\Rightarrow \neg(\forall x\ P(x))$ is true.

Similarly $\neg(\exists x\, Q(x)) \equiv \forall x\, \neg Q(x)$

if there does not exist an x s.t. $Q(x)$ then every x gives $\neg Q(x)$.

Ex: Negate the sentence "Some one will fail this class" where $P(x)$ = pass class

Some one will fail this class $\Rightarrow \exists x\, \neg P(x)$

$\neg(\exists x\, \neg P(x)) \equiv \forall x\, \neg\neg \cancel{P}(x) \equiv \forall x\, P(x)$

"Every one will pass this class.

Now onto fun stuff!

First, puzzles: Suppose there are two people and they either only tell the truth or only lie, independently. A telling the truth has no implication on B. They then say:

A: "Exactly one of us is lying"

B: "At least one of us is truthful"

What can you conclude about the truthfulness of A/B?

Lets do this rigorously:

A truthteller?	B truthteller?	A told truth?	~~A~~B told truth?	viable
T	T	F	T	NO.
T	F	T	T	NO,
F	T	T	T	NO
F	F	F	F	Yes,