proof examples

Ex: prove that if x & y are rational then xy is rational.

pf: x, y rational $\Rightarrow x = \frac{p}{q}$ $y = \frac{r}{s}$ p, q, r, s integers $q, s \neq 0$.

Then $xy = \frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$ pr, qs integers

$qs \neq 0 \Rightarrow xy$ rational. $\square$.

Ex: prove $x \cdot 0 = 0$ for any real number x.

pf: $x \cdot 0 + 0 = x \cdot 0 = x(0+0) = x \cdot 0 + x \cdot 0$

$\Rightarrow x \cdot 0 = 0$

$\square$

Ex: Let $A = \frac{S_1 + S_2 + \cdots + S_n}{n}$ for some n numbers S_i.

prove $\exists i$ s.t. $S_i \leq A$.

pf By contradiction: Suppose $\nexists i$ s.t. $S_i \leq A \Rightarrow \forall i \; S_i > A$

Then $A = \frac{S_1 + S_2 + \cdots + S_n}{n} > \frac{A + A + \cdots + A}{n} = \frac{nA}{n} = A$

So $A > A$ $\rightarrow\leftarrow$. Thus $\exists i$ s.t. $S_i \leq A$

$\square$.

Ex: There are infinitely many prime numbers.

Pf By Contradiction: Assume there are only finitely many prime numbers. Then we can list them: $p_1, p_2, \dots, p_n$

Now consider the integer $P = p_1 \cdot p_2 \cdot p_3 \cdots p_n + 1$

does $p_i \mid P$? No, reminder 1. so P is not divisible by any number other than itself & 1. $p_i \neq P$ for any i so P is a prime not in our list $\rightarrow\leftarrow$ to list of all primes.

Thus there are infinitely many primes. $\square$

Ex: prove or disprove: There exist rational numbers a, b s.t. a^b is rational.

pf: $a = 2 = b$. $\square$

Ex: prove or disprove: There exist rational numbers a, b S.t. a^b is irrational.

pf $a = 2$ $b = \frac{1}{2}$ $\square$

Ex: prove or disprove: There exist irrational numbers a, b s.t. a^b is rational.

pf: Case 1: $a = b = \sqrt{2}$ If $\sqrt{2}^{\sqrt{2}}$ is rational done.

Case 2: $a = \sqrt{2}^{\sqrt{2}}$ $b = \sqrt{2}$ then $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^2 = 2$. $\square$

Ex: prove that $2m^2+4n^2-1 = 2(m+n)$ has no solution in positive integers.

Pf $\Rightarrow$ $2m^2-2m+4n^2-2n = 1 \Rightarrow m^2-m+2n^2-n = \frac{1}{2}$

No addition of integers $= \frac{1}{2}$

$\square$.

Ex: prove $m^3+2n^2=36$ has no solution in positive integers

Note: $1 \le m \le 3$ Since $4^3=64$.

$1 \le n \le 4$ Since $2\cdot 5^2 = 50$.

Cases: If $m=1,3$: Then $m^3+2n^2=36 \Rightarrow 2n^2 = \{35, 9\}$

So an even $=$ odd $\rightarrow\leftarrow$.

If $m=2$. $m^3+2n^2=36 \Rightarrow 2n^2=28 \Rightarrow n^2=14$.

no integer squares to 14.

Thus no assignment of positive integers solves the eqn.

$\square$.

Ex: Define $Sgn(x) = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -1 & \text{if } x<0 \end{cases}$ prove $|x| = sign(x)\cdot x$.

for all real x.

Pf: Cases: Case 1, $x>0$: Then $|x| = x = 1\cdot x = Sgn(x)\cdot x$.

case 2; $x=0$: $|x| = 0 = 0\cdot x = Sgn(x)\cdot x$.

case 3, $x<0$: $|x| = -x = -1\cdot x = Sgn(x)\cdot x$.

$\square$.

Ex: triangle inequality: If x, y are real then $|x|+|y| \geq |x+y|$.

Pf:

$$-|x| \leq x \leq |x|$$
$$-|y| \leq y \leq |y|$$
$$\Rightarrow -(|x|+|y|) \leq x+y \leq |x|+|y|$$

Note if $-a \leq b \leq a \Rightarrow |b| \leq |a|$
worst case is $b = a, -a$

So $|x+y| \leq |x|+|y|$.

$\square$.

Ex: Prove there are 100 consecutive integers none of which are a perfect square.

Pf: Choose $10{,}001 - 10{,}101$.

$100^2 = 10{,}000$, $101^2 = 10{,}201$ So no number in between can be a perfect square.

$\square$.

Ex: Prove that either $2 \cdot 10^{500} + 15$ or $2 \cdot 10^{500} + 16$ is not a perfect square.

Pf: Suppose for a contradiction, they both are: then $2 \cdot 10^{500} + 15 = n^2$ & $2 \cdot 10^{500} + 16 = k^2$ for some integers n, k.

Then $n^2 - k^2 = 1 \Rightarrow (n-k)(n+k) = 1$ $n-k, n+k$ integers

$\Rightarrow n-k = n+k \Rightarrow 0 = 2k \Rightarrow k = 0 \rightarrow\leftarrow 0^2 \neq 2 \cdot 10^{500} + 16$.

Thus at least one of $2 \cdot 10^{500} + 15$ & $2 \cdot 10^{500} + 16$ must not be a perfect square.

$\square$

Ex: prove $1+3+5+\cdots+(2n-1)=n^2$

Pf: By Induction; Base Case If $n=1$ then $1 = n^2$ ✓.

IH: Assume $1+3+\cdots+(2k-1)=k^2$ for some $k\geq 1$ we show

$1+3+5+\cdots+(2k-1)+(2(k+1)-1)=(k+1)^2$

$$\overset{IH}{=} k^2+2k+2-1$$
$$= k^2+2k+1$$
$$= (k+1)^2.$$

This closes the induction. □

Ex: prove $1\cdot 2+2\cdot 3+\cdots+n(n+1)=\frac{n(n+1)(n+2)}{3}$

Pf By Induction. Base Case; $n=1$, $1\cdot 2=\frac{1\cdot 2\cdot 3}{3}$ ✓

IH: Assume eqn holds for some $k\geq 1$ we show it holds for $k+1$. That is, we show $1\cdot 2+\cdots+k(k+1)+(k+1)(k+2)$
$=\frac{(k+1)(k+2)(k+3)}{3}$

$$LHS \overset{IH}{=} \frac{k(k+1)(k+2)}{3}+(k+1)(k+2)$$
$$=\frac{k(k+1)(k+2)+3(k+1)(k+2)}{3}$$
$$=\frac{(k+1)(k+2)(k+3)}{3}$$

This closes the induction. □

Ex. prove $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Pf: By Induction: Base case, $n=1$: $1^2 = \frac{1 \cdot 2 \cdot 3}{6}$ ✓

IH: Assume identity holds for some $k \geq 1$ we show it holds for $k+1$:

$$1^2 + 2^2 + \cdots + k^2 + (k+1)^2 \overset{IH}{=} \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)(k+1)}{6}$$

$$= \frac{(k+1)(k(2k+1) + 6(k+1))}{6}$$

$$= \frac{(k+1)(2k^2 + k + 6k + 6)}{6}$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

This closes the induction. $\square$

Ex: Find a formula for

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}$$

and prove it holds for all positive integers n.

$\frac{1}{1 \cdot 2} = \frac{1}{2}$ $\qquad \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$

$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{6}{12} + \frac{2}{12} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4}$

$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{30}{60} + \frac{10}{60} + \frac{5}{60} + \frac{3}{60}$

$= \frac{48}{60} = \frac{12}{15} = \frac{4}{5}$

Conjecture $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\dots+\frac{1}{n(n+1)}=\frac{n}{n+1}$.

Pf By Induction, Base case $n=1$, $\frac{1}{1\cdot 2}=\frac{1}{2}$ ✓

IH: Assume identity holds for some $k\geq 1$ we show it holds for $k+1$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\dots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{(k+1)^2}{(k+1)(k+2)}$$

$$=\frac{(k+1)}{k+2}$$. This closes the induction.

□.

Ex: Show that postage of 4 cents or more can be achieved using only 2-cent & 5-cent stamps,

Pf By Strong Induction. Base cases: $n=4, 5$

We can make 4 cents with 2 2-cent stamps, we can make 5 cents from 1 5 cent stamp,

IH: Assume $n\geq 6$ and for each k $4\leq k<n$ we can make postage using only 2 & 5 cent stamps, we show we can make n cent postage. By IH we can make $n-2$ cent postage. Then add one 2cent stamp.

□.

Ex: Define a sequence $c_1, c_2, \ldots$ by

$$c_1 = 0, \quad c_n = c_{\lfloor \frac{n}{2} \rfloor} + n^2 \text{ for all } n > 1$$

Prove $c_n < 4n^2$ for all $n > 1$

work space: ~~$n=1$ $c_1 = 0 < 4$~~ ✓

$n=2$ $c_2 = c_1 + 2^2 = 0+4 < 4 \cdot 4$ ✓

$n=3$ $c_3 = c_1 + 3^2 = 0+9 < 4 \cdot 9$ ✓

$n=4$ $c_4 = c_2 + 4^2 = 4+16 < 4 \cdot 16$ ✓

How far to go?

we need to claim holds for k $n_0 \le k < n$ where $n_0 = 2$

we will assume true for $k = \lfloor \frac{n}{2} \rfloor \Rightarrow 2 \le \lfloor \frac{n}{2} \rfloor < n$

$n=2$ $\lfloor \frac{2}{2} \rfloor < 2$ $\quad n=3$ $\lfloor \frac{3}{2} \rfloor < 2$

$n=4$ $\lfloor \frac{4}{2} \rfloor = 2$ $\quad$ So $n = 2, 3$ needed base case,

$n=4$ we can use our needed I.H.

Pf By Induction. Base Case $n = 2, 3$ (above).

IH Assume $n \ge 4$ & equality holds for all k $2 \le k < n$

we show it holds for n.

$c_n = c_{\lfloor \frac{n}{2} \rfloor} + n^2$ $\quad$ (note $\lfloor \frac{n}{2} \rfloor < n$ & $n \ge 4 \Rightarrow \lfloor \frac{n}{2} \rfloor \ge 2$)

$$\overset{IH}{<} 4\left(\lfloor \tfrac{n}{2} \rfloor\right)^2 + n^2$$

$$\le 4 \frac{n^2}{4} + n^2$$

$$= n^2 + n^2$$

$$= 2n^2$$

$$< 4n^2.$$

This closes the induction. $\square$