Probability Theory

Given a sample space S we can talk about probability of particular outcomes $0 \le p(s) \le 1$ $\forall s \in S$. But in particular

$$\boxed{\sum_{s \in S} p(s) = 1}$$ ← the result of the experiment must give something in S.

The function p that assigns the probability of each outcome gives a probability distribution over S.

Ex What probabilities should we assign to the outcomes H & T when a fair coin is flipped? What probabilities should we assign when the coin is biased so heads comes up twice as often as tails?

(1) $p(H) + p(T) = 1$ $\quad p(H) = p(T) \Rightarrow 2p(H) = 1 \Rightarrow p(H) = p(T) = \frac{1}{2}$.

(2) $p(H) + p(T) = 1$ $\quad p(H) = 2p(T) \Rightarrow 3p(T) = 1 \Rightarrow p(T) = \frac{1}{3}$

$p(H) = \frac{2}{3}$.

Def: Suppose $|S| = n$. The uniform distribution assigns $\frac{1}{n}$ probability to each outcome in S.

Def: The probability of an Event $E \subseteq S$ is the sum of probability of outcomes in E

$$p(E) = \sum_{s \in E} p(s)$$

Ex: Suppose a die is biased to roll 3 twice as often as any other number, but the other five values are equally likely. What is the probability of rolling an odd?

$P(1) + P(2) + \dots + P(6) = 1$ $\qquad$ $P(3) = 2P(1)$

$\Rightarrow 5P(1) + P(3) = 1 \Rightarrow 7P(1) = 1 \Rightarrow P(1) = \frac{1}{7}$

& $P(3) = \frac{2}{7}$ $\qquad$ $1 + 1 + 2 + 1 + 1 + 1 = 7$ ✓

$P(\text{odd}) = P(\{1, 3, 5\})$ $\qquad$ $P(1) + P(3) + P(5) = \frac{1}{7} + \frac{2}{7} + \frac{1}{7}$

$= \frac{4}{7}.$

Conditional probability: Imagine flipping a fair coin 3 times $\Rightarrow$ 8 (2^3) possible outcomes. Suppose we also know the first flip comes up tails. Given this info. what is the probability of an odd number of tails.

This new info has changed our sample space instead of 8 possible outcomes there are only 4: TTT TTH THT THH.

Only TTT, THH give an odd number of tails so

$P(\text{odd, given first tail}) = \frac{1}{2}$

More generally this is stated the probability of E given F

Def: Let E, F be event $P(F) > 0$ The conditional probability of E given F is

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

Ex: A bit string of length 4 is generated at random, uniformly. What is the probability the resulting string contains consecutive 0's given the first bit is 0?

$$P(\text{consecutive 0's} \mid \text{first bit 0}) = \frac{P(\text{both})}{P(\text{first bit 0})}$$

$P(\text{first bit } 0) = \frac{1}{2}$ $\quad P(\text{both}) = \frac{5}{16}$

both = {0000, 0001, 0010, 0011, 0100}

$$= \frac{5/16}{1/2} = \frac{10}{16} = \frac{5}{8}.$$

This is complicated to do in general if we need to list all possible outcomes: need a formula!

<u>Def</u>: The events E & F are <u>independent</u> iff $P(E \cap F) = P(E)P(F)$

i.e. one does not affect the other.

<u>Ex</u>: We generate a bit string of length 4. Let E = string starts with 1

F = string has an even number of 1s? Are these independent?

$E \cap F = \{1111, 1100, 1010, 1001\}$ $\quad P(E \cap F) = \frac{4}{16} = 1/4$

E = {1111, 1110, 1101, 1011, 1010, 1001, 1000, 1100}

$P(E) = \frac{8}{16} = \frac{1}{2}$

F = {0000, 0011, 0101, 1001, 0110, 1010, 1100, 1111}

$P(F) = \frac{8}{16} = \frac{1}{2}$ $\quad P(E) \cdot P(F) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$\Rightarrow$ E & F are independent

Ex: Suppose a family is going to have two children. A boy & a girl are equally likely. Let E = they have two boys

F = they have at least one boy. Are E & F independent?

$E = \{BB\}$ $P(E) = \frac{1}{4}$ $F = \{BG, GB, BB\}$

$P(F) = \frac{3}{4}$ $P(E) \cdot P(F) = \frac{3}{16}$

$E \cap F = \{BB\} \Rightarrow P(E \cap F) = \frac{1}{4} \neq$ so not independent.

This should make sense conceptually. Event F occurring makes E more likely. So F influences E $\Rightarrow$ not independent.

Ex: Flipping a coin 2 times. getting H does not affect next flip.

$\Rightarrow$ independent

Ex: Drawing cards from a deck. drawing an Ace on the first card drastically affects the odds of drawing another ace on second draw

$\Rightarrow$ not independent.