Probability

Def 1. An experiment is a procedure that yields possible outcomes.

2. The sample space of an experiment is the set of possible outcomes.

3. An event is a subset of the sample space

4. If S is non-empty sample space of equally likely outcomes & $E \subseteq S$ the probability of E is $P(E) = \frac{|E|}{|S|}$

Definition 4 tells us: $0 \le |E| \le |S|$ (since $E \subseteq S$) $\Rightarrow$

$$0 \le P(E) \le 1$$

The probability of any event is between 0 & 1.

Ex: Two fair dice are rolled. what is the probability that the sum of the numbers on the dice is 10?

To answer this we must count 2 quantities

1. the number of all possible dice rolls (S = all possible dice rolls)
   $\underline{6} \cdot \underline{6} = 36$
2. The number of all dicerolls that add to 10. (E = all dicerolls that add to 10)

no slick way here just have to find them

If D1 rolls 1, 2, 3 then can't sum ~~to~~ to 10

$$\left.\begin{array}{l} D1=4 \Rightarrow D2=6 \\ D1=5 \Rightarrow D2=5 \\ D1=6 \Rightarrow D2=4 \end{array}\right\} \Rightarrow 3 \text{ outcomes result in our event.}$$

Thus the probability that the dice rolls sum to 10 is $\frac{3}{36} = \frac{1}{12}$

Ex: five micro processors are randomly selected from a lot of 1000 microprocessors among which 20 are defective. Find the probability that no defective microprocessors are detected.

1. total number of ways to select 5 microprocessor

$$\binom{1000}{5}$$

2. Ways to select ~~15~~ 5 non-defective microprocessors:

$$\binom{980}{5}$$

$\Rightarrow$ probability of selecting all non-defective processors

$$P(E) = \frac{\binom{980}{5}}{\binom{1000}{5}} = 0.9037\ldots$$

Ex: In a lottery game to win the grand prize the contestant must match 6 distinct numbers, in any order, among the numbers 1-52, randomly drawn. What is the probability of winning?

1. How many ways can we chose 6 numbers?

$$\binom{52}{6}$$

2. How many winning choices are there?

1 - only one because does not include order either.

$$\text{so } P(\text{winning}) = \frac{1}{\binom{52}{6}} = 0.000000049$$

Theorem: Let E be an event in a sample space S. The probability of $\bar{E} = S \setminus E$ is given by

$$P(\bar{E}) = 1 - P(E)$$

Ex: A sequence of 10 bits is randomly generated. What is the probability that at least one bit is 0?

We can try to count all ways at least one bit is 0 but thats a lot of work. Instead lets count the number of bit strings with no 0's, only 1! Thus

$$P(\text{at least one zero}) = 1 - P(\text{no zeros})$$
$$= 1 - \frac{1}{2^{10}}$$
$$= \frac{1023}{1024}$$

Ex: Birthday problem: Find the probability that among n people at least 2 have the same birthdate (month & day). Assume all dates are equally likely & ignore Feb. 29th

We can count the total number of possible birthdays easily.

365^n - each persons birthday is randomly selected among each of the days.

But counting the ways that at least 2 people share a birthday is complicated. You'd need to look at all possible number of ppl who could share $2 - n$ choose which of them will share, choose the date they'll share then count the ways the remaining people won't share with anyone.

Its much easier to count how many ways noone shares birthdays.

$\underline{365}$ $\underline{364}$ $\underline{363}$ $\underline{362}$ ... $\underline{365 - n + 1}$

So $P(\text{no shared birthdays}) = \dfrac{365 \cdot 364 \cdots (365-n+1)}{365^n}$

$\Rightarrow$ $P\left(\begin{array}{l}\text{at least 2 ppl share} \\ \text{bday}\end{array}\right) = 1 - P(\text{no shared bday})$

when $n=22$ this probability is 0.475695

$n=23$ 0.507297 ← So in a group of 23 or more ppl its more likely 2 ppl share a bday than not

whole division $n=36$ 0.598240 ← so pretty likely we'd get a match.

The reason this is so large so small is due to the number of groups that can be formed. with more people the number of pairs of ppl grows very quickly!

<u>Theorem</u>: Let E_1, E_2 be events in the sample space S. Then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2).$$

<u>Ex</u>: What is the probability that a randomly selected integer not exceeding 100 is divisible by either 2 or 5?

Let E_1 = integer is divisible by 2 $|E_1| = 50$

E_2 = integer is divisible by 5. $|E_2| = 20$

$E_1 \cup E_2$ = integer divisible by either 2 or 5. $|E_1 \cup E_2| = ?$

$E_1 \cap E_2$ = integer divisible by 2 & 5 $|E_1 \cap E_2| = 10$

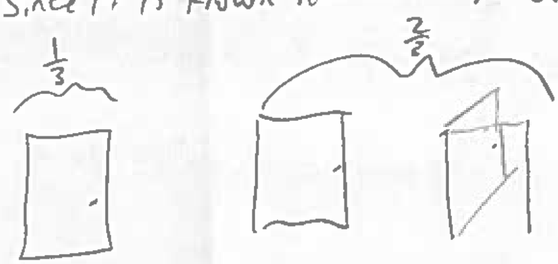
So $P(E_1 \cup E_2) = \dfrac{50}{100} + \dfrac{20}{100} - \dfrac{10}{100} = \dfrac{3}{5}$.

Probabalistic reasoning can be hard to follow sometimes.

<u>Ex</u>: <u>The Monty Hall Problem</u>: Suppose you're a game show contestant. There are 3 doors you will choose to open one, one door has a big prize behind it the other two are losers. Once you select a door the host will choose one of the un-selected doors that is a loser & open it. Then he will ask whether you would like to switch your door. Should you?

The probability that you select the correct door initially is $\frac{1}{3}$

The probability one of the other two doors is correct is $\frac{2}{3}$. The probability that your door was correct does not change when the host opens a door. Since it is known he will always open a bad door.



Thus switching gives you a $\frac{2}{3}$ chance of winning.

Imagine 100 doors, you choose one, host opens 98 then asks if you want to switch. Which is more likely you correctly guessed a $\frac{1}{100}$ choice or the host avoided opening the door with the prize?