Predicates & Quantifiers

Propositional Logic is lacking. Given that

" all even integers are divisible by 2"

We can't logically prove "4 is divisible by 2"

We need predicates : essentially variables.

Ex: $x > 3$ $\qquad$ x is divisible by 2

Def: Let $P(x)$ be a statement involving the variable x

We call P a propositional function or Predicate if

for each x $P(x)$ is a proposition.

Ex $P(x)$ = "$x > 3$" - not a proposition

but for any value of x it is!

What are the truth values of $P(4)$ & $P(2)$?

T F

Ex: $Q(n) = n^2 + 2n$ is an odd integer.

Is this a proposition? No it's a predicate.

value of $Q(2) = T$

$Q(3) = F$

This on its own is not enough. In math (and CS) we don't get much use from this. We don't get use from claims such as $Q(3)$ is true etc.

We need more generality:

Quantifiers:

Quantification allows us to express truth values for ranges such as all, some, many, none, & few.

Def: Universal Quantifier: Many math statements assert something is true for all inputs in a domain such as $P(x)$ is true for all x in domain.

We use $\forall$ for this e.g. $\forall x\, P(x)$ read as for all x, $P(x)$ or $P(x)$ for every x. The claim is true if $P(x)$ is true for every valid x in the domain. It is false if some x in the domain make $P(x)$ false. This x is a Counter example

Ex: $\forall x\ (x^2 \geq 0)$ for x in real numbers. Here $P(x) = x^2 \geq 0$. our domain is real numbers.

It is true if for every real number x, $x^2 \geq 0$ & false if there is at least one x_0 in the real numbers such that $x^2 < 0$.

this is true.

Remarks: 1. We usually assume our domains are not empty.

$\forall x P(x)$ requires that $P(x)$ be true for all x in domain. if there is no x in domain $\forall x P(x)$ is vacuously true

2. When discussing this avoid using "any". In English any is acceptable to use for many or lots. But $\forall$ means for every one, for all of them, for each thing etc.

Ex: In some cases we can think of $\forall$ being a Conjunction:

What is the truth value of $\forall x\, P(x)$ where $P(x) = x^2 < 10$ and our domain is positive integers less than 5.

Then $\forall x\, P(x) = P(1) \wedge P(2) \wedge P(3) \wedge P(4)$

$P(4) = F$ So $\forall x\, P(x)$ is False.

Existensial Quantifier:

Def: The existensial quantifier of $P(x)$ is the statement

"There exists an element x in the domain such that $P(x)$"

This is denoted $\exists x\, P(x)$

This can be stated as there is an x s.t. or there exists x s.t.

or for some x $P(x)$

Ex: $\exists x \left(\frac{x}{x^2+1} = \frac{2}{5} \right)$ So $P(x) : \frac{x}{x^2+1} = \frac{2}{5}$

is $P(x)$ a proposition? No! it is a predicate.

$\exists x\, P(x)$ is a proposition, it is true $x = 2 \Rightarrow P(2) = T$.

Only need to find one x.

Ex: ~~Consider~~ Show $\exists x\, Q(x)$ where $Q(x) : x = x+1$

What is the truth value of $\exists x\, Q(x)$?

False, there is no number x where $x = x+1$ So we cannot find an x to satisfy $Q(x)$.

Notice: to disprove existential quantifiers we must show it is always false, on its domain. More on this next time.

Ex: What is the truth value of $\exists x\, P(x)$ for $P(x) : x^2 > 10$ on domain of positive integers less than 5.

This is equivalent to $P(1) \vee P(2) \vee P(3) \vee P(4)$

we only need one to be true. Indeed $P(4) = T$

So $\exists x\, P(x)$ is true on this domain.