Pigeon hole principle

Suppose a flock of 20 pigeons wants to fly into a collection of 19 pigeonholes. There must be atleast one hole with atleast 2 pigeons!

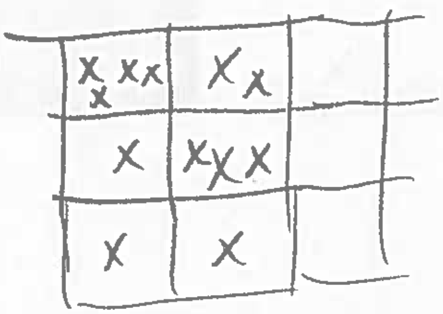
Theorem: If $k \in \mathbb{N}$ & $k+1$ objects are placed in k boxes then atleast one box has two or more objects

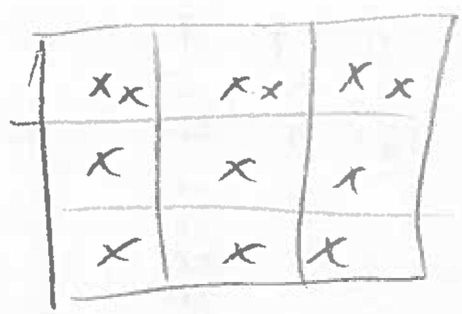
pf: By contradiction: If no box has more than one object then we have less than $k+1$ objects $\rightarrow\leftarrow$

$\square$

This is a very loose statement!

e.g. 12 x's & 9 boxes





Don't know the distribution of objects just atleast one box has more than one objects.

Ex: In any group of 27 English words at least two words begin with the same letter.

Theorem: Generalized Pigeonhole: If N objects are placed in k boxes there is at least one box containing $\lceil \frac{N}{k} \rceil$ objects.

Ex: In this class there are 39 people enrolled. There are 5 possible grades A, B, C, D, F. How many people are guaranteed to end up with the same grade?

$$\left\lceil \frac{39}{5} \right\rceil = \frac{40}{5} = 8$$

Can think of this as spreading out as far as possible

A	B	C	D	F
1	2	3	4	5
6	7	8	9	10
⋮				15
				20
				25
				30
				35
			39	

Ex: ppl. in class w/ birth month.

Ex: How many cards from a standard 52-card deck must be drawn to guarantee 3 cards of the same suit are chosen?

$\lceil \frac{x}{4} \rceil = 3$ if $x = 8$ $\frac{8}{4} = 2$ so $\lceil \frac{9}{4} \rceil = 3$

How many to guarantee 3 hearts? ⟵ Whats different about this question? Not Pigeonhole principle.

worst case first 39 cards are all spades, clubs, diamonds

so 42 guarantees 3 hearts

Ex: US phone numbers are of this form NXX - NXX - XXXX where $N \in \{2, 3, \ldots, 9\}$ $X \in \{0, \ldots, 9\}$ How many area codes are needed to ensure 25 million phones have unique numbers?

first examine last 7 digits

NXX - XXXX

$8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$

$= 8 \cdot 10^6 = 8$ million numbers

so 8 million numbers per area code $\Rightarrow$ $\lceil \frac{25}{8} \rceil = 4$ area codes are needed.

Sometimes we must be clever!

Ex: During a month with 30 days a baseball team plays at least one game per day but no more than 45 games in 30 days. Show there must be some number of consecutive days where the team plays 14 games exactly.

Chose a_j to be the number of games played on or before day j

Then a_j is increasing & $1 \le a_j \le 45$. Indeed $a_1+14, a_2+14, \ldots, a_{30}+14$

is also increasing with $15 \le a_j \le 59$

$a_1, a_2, \ldots, a_{30}, a_1+14, \ldots, a_{30}+14$ is 60 numbers

all less than or equal to 59. $\Rightarrow$ 2 integers are equal.

a_j are all distinct (increasing) $\Rightarrow \exists\, i, j$ s.t. $a_i = a_j + 14$

Thus 14 games must be played from day $j+1$ to day i.

Ramsey theory: Assume in a group of 6 people each pair of individuals are either friends or enemies. Show there are either 3 mutual friends or 3 mutual enemies.

Pf: Choose one person, label them A. There are 5 remaining ppl

each person is either an enemy or friend $\Rightarrow$ 2 groups

$\Rightarrow$ One group must contain atleast 3 people $\lceil \frac{5}{2} \rceil = 3$

AWLOG B, C, D are all friends with A. If any two of them are friends, then with A they form mutual friends. If none do they form 3 mutual enemies.

$\square$

whole area of math $R(m,n)$ = minimum number of ppl at party s.t. there are m mutual friend or n mutual enemies. we demonstrated $R(3,3) \le 6$.