Partial Ordering!

Equivalence relations are very strong & completely break up sets, give us info on which elements are the same etc. There are other properties relations can have to give info too.

Def: A relation R on S is called a partial ordering if it is reflexive, anti-symmetric & transitive. (S, R) is then called a poset (partially ordered set).

Ex: $S = \mathbb{Z}$ $R = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : a \leq b\}$

reflexive: $a \leq a$ ✓

anti-sym: If $a \leq b$ & $b \leq a \Rightarrow a = b$ ✓

transitive: $a \leq b$ & $b \leq c \Rightarrow a \leq b \leq c \Rightarrow a \leq c$ ✓

Thus $(\mathbb{Z}, \geq)$ is a poset.

Ex: Let S = set of all people, xRy iff x is older than y.

Not a partial ordering, why? Not reflexive.

However it is anti symmetric. ← if statement: if aRb & $bRa \Rightarrow a = b$

we fail the hypothesis: if x older than y then y not older than x!

A partial ordering says we can follow the chain in one direction only.

However not everything need be related.

For partial orderings the symbol $\preceq$ is often used.

Def: The elements a, b in the poset $(S, \preceq)$ are <u>comparable</u> if $a \preceq b$ or $b \preceq a$. Otherwise they are incomparable.

In our first example any two numbers are comparable. In the second two people are incomparable if if they are born at the same time.

Ex: $(S, \subseteq)$ is a poset for S any collection of sets, but many are incomparable. $S = \{\{1,2\}, \{3,4\}, \{1,2,3,4\}\}$

$$\{1,2,3,4\}$$

$$\subseteq\!/ \qquad\qquad \cup\!\!\!\setminus$$

$$\{1,2\} \qquad\qquad \{3,4\}$$

but $\{1,2\}$ & $\{3,4\}$ are incomparable

Def: If any two elements are comparable $(S, \preceq)$ in the poset it is called a <u>total ordering</u> or <u>totally ordered</u>

Def: $(S, \preceq)$ is <u>well ordered</u> if it is totally ordered and there is a least (first) element.

Theorem Induction can be done over any <u>well-ordered</u> set.