Infinite Sets

Recall we define $|A|$ to be the Cardinality of A, the number of elements in A.

Proposition: If $f: A \to B$ is one-to-one then $|A| \le |B|$.

Pf: f Being one-to-one $\Rightarrow$ each element of $f(A)$ has exactly one preimage.

$\Rightarrow |A| = |f(A)|$ & $f(A) \subseteq B \to |f(A)| \le |B|$ or $|A| \le |B|$.

$\square$

Def: A set has an infinite number of elements if it does not have a finite number of elements.

We now split infinity into two groups, Countable & uncountable.

Def: A set that is finite or has the same cardinality of $\mathbb{N}$ is countable

A set that is not countable is uncountable.

Note: two sets have the same Cardinality iff $\exists$ a bijection between the sets.

Ex $A: \{1, 2, 3\}$ $B: \{a, b, c\}$ have the same cardinality

b/c
$1 \mapsto a$
$2 \mapsto b$
$3 \mapsto c$
is a bijection.

Ex $C: \{4, 5, 6\}$ $D: \{d, e\}$ have different cardinalities,

no map can be injective.

Seems unnecessarily complicated, but its necessary for infinite sets.

Ex: Show the set of odd positive integers is countable (same cardinality as $\mathbb{N}$).

Pf: Consider the fcn; $f: \mathbb{N} \to O$: set of odd positive integers

$f(n) = 2n+1$

inj: if $n \neq m$ then $2n \neq 2m \Rightarrow 2n+1 \neq 2m+1$ ✓

surj: If $k \in O$ by def $k = 2m+1$ some $m \in \mathbb{Z}$ but m must be ≥ 0

if $m < 0 \Rightarrow 2m < 0$ so $2m+1 < 1$ So $m \in \mathbb{N}$. ✓

Thus f is a bijection, So O is countably infinite.

$\square$

Ex: Hilberts Hotel: Imagine a hotel with countably infinite rooms, one for each number $n \in \mathbb{N}$. Now assume every room is occupied.

Now a new person arrives & asks for a room. Can you accomodate them?

Yes! ask everyone to move to their room $+1$. If your in room k go to room $k+1$. new person gets room 0.

Now countably infinitely many ppl arrive, lets label them $-1, -2, -3, \ldots$ Can we accomodate them?

Yes! ask person in room k to move to room $2k$ & then the new group gets room $2|m|-1$ So $-1 \to 1$ $-2 \to 3$ etc.

This is the proof that $|\mathbb{Z}| = |\mathbb{N}|$, crazy huh? It gets crazier!

<u>Ex</u>: The set of rationals $\mathbb{Q}$ is countable.

<u>Pf</u>: This is a bit more hand wavy. We are going to construct a grid:

$$\begin{array}{cccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \cdots \\
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \cdots \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

This will list almost every rational number (no negatives), & a few others. For this $\frac{2}{2} \notin \mathbb{Q}$ only reduced numbers are.

To be countable we need a method to assign what the first element is, the second etc. Call $\frac{1}{1}$ the first. then go down a row, $\frac{1}{2}$ is second then diagonal up: $\frac{2}{1}$ is the third then right $\frac{3}{1}$ is the fourth then diagonal down, skip $\frac{2}{2}$ b/c we've seen $\frac{1}{1}$ so $\frac{1}{3}$ is the fifth one. This will hit every number in grid & hits no number more than once $\Rightarrow$ bijection

$\Rightarrow$ Positive rationals are countable. Then do the same trick as Hilberts hotel to get negatives too!

D.

Any Countable set is said to be well-ordered. A well ordered set is one which has a first element & given an element you know what the next element is.

e.g. $\mathbb{N}$ is well ordered. 0 is first, given k k+1 follows.

Any countable set is well ordered b/c $\exists$ f a bijection $f: \mathbb{N} \to S$

So $f(0)$ is the first element of S & given $s \in S$ $s = f(k)$ so $f(k+1)$ is the next.

Ex In pos. rats. $\frac{1}{1}$ is first then $\frac{1}{2}$ is next. Given $\frac{m}{n}$ we can find which element comes next (a bunch of work in this case).

It may seem that all sets are countable if even $\mathbb{Q}$ is. Why even have uncountable. Turns out $\mathbb{R}$ is uncountable.

Ex: $\mathbb{R}$ is uncountable

Pf: By Contradiction, assume $\mathbb{R}$ is Countable. Then the subset of reals between 0 & 1 is also countable. Put all numbers between 0 & 1 in some order (we can do this b/c countable $\Rightarrow$ well ordered). So we have

$r_1 = 0.d_{11}d_{12}d_{13}\ldots$ $\qquad d_{ij} \in \{0,1,2,3,4,5,6,7,8,9\}$

$r_2 = 0.d_{21}d_{22}d_{23}\ldots$

$r_3 = 0.d_{31}d_{32}d_{33}$

$\vdots$

$r_n = 0.d_{n1}d_{n2}d_{n3}\ldots$

$\vdots$

Now construct

$r = 0.d_1 d_2 d_3 \ldots$

where $d_i = \begin{cases} 4 & \text{if } d_{ii} \neq 4 \\ 5 & \text{if } d_{ii} = 4. \end{cases}$

e.g. 0.23794102 $\qquad$ $r = 4544\ldots$

0.44590138

0.09118764

0.80553900

then by construction r is not on our list but is between 0 & 1.

notice r disagrees w/ r_i at spot r_{ii} so $r \neq r_{ii}$ $\forall i \in \mathbb{N}$

Thus $(0,1)$ is not Countable $\Rightarrow \mathbb{R}$ is not Countable $\Rightarrow \mathbb{R}$ is uncountable.

$\square$.

Note by convention $|\mathbb{R}| = |P(\mathbb{N})|$

This proof secretly relies on $1 = 0.999\ldots$

Because we need every real number to be distinct from others.

So lets prove this LOTS of WORK, Ready?!

Ex: $1 = 0.999$

Pf Let $x = 0.999\ldots$

$$\begin{aligned} 10x &= 9.999\ldots \\ &= 9 + 0.9999\ldots \\ &= 9 + x \\ 9x &= 9 \\ x &= 1 \end{aligned}$$

or

$$\frac{1}{9} = 0.11111\ldots$$

$$\Rightarrow 1 = \frac{9}{9} = 0.99999$$

There are analytic proofs as well that demonstrate there is no number between 1 & $0.999\ldots$

Matrices:

Def: A matrix is a rectangular array of numbers. A matrix with m rows & n columns is called an $m \times n$ matrix. If $m = n$ then the matrix is square.

Ex: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ is a 3×2 matrix.