<u>Induction</u>: Some things we want to prove are about all integers.

Some of these proofs are <u>not</u> easy to prove with our given methods.

Enter Induction.

<u>Heads up</u>: This is more complicated than any previous method. Every class there is someone who doesn't get it or just doesn't believe induction, that's fine. It always happens, don't be upset we'll get through it.

<u>Ex</u>: Suppose we have infinitely many blocks labeled 1, 2, 3, ...
& at least some of the blocks are marked with "x".
Additionally we know, the first block is marked &
for any block, if the previous block is marked then so is this block.

what can we conclude?

All blocks are marked. why? well the first is marked ⇒
Second is marked ⇒ third is marked ⇒ ...
for the nth block, the n-1 th is marked (assumption) ⇒ nth is marked.

<u>Induction consists of 2 steps</u>: a <u>base step</u>: show a particular instance $P(1)$ is true. & an <u>inductive step</u>: where we show in general if $P(k)$ then $P(k+1)$.

what does this sound similar to in CS? Recursion.

Ex. Show $p(n)$ is true for all positive integers n.

Pf: Base step. $P(1)$ is true

IH: Assume $P(k)$ is true $\forall k \leq n$.

we show $p(n+1)$ is true.

$\square$

Logically this is expressed as

$$\Big(P(1) \wedge \big(\forall k\, (P(k) \to P(k+1))\big)\Big) \longrightarrow \forall n\, P(n).$$

Note: we are not assuming $P(k)$ is true for all k

that is what we prove we are showing if we assume $p(k)$ then $p(k+1)$ is a true statement. And showing $\exists k$ s.t. $P(k)$ is true Together this gives $P(k+1)$, $P(k+2)$, $P(k+3)$, etc...

This method is valid, but is often unsatisfying. It does nothing to explain why something is true, only that it is.

Ex: Show if n is a positive integer, then

$$1+2+\cdots+n = \frac{n(n+1)}{2}$$

Pf By Induction,

Base step: $n=1$, $1 = \frac{1(2)}{2} = 1$ ✓

IH: Suppose our hypothesis holds for some k. that is

$$1+2+\cdots+k = \frac{k(k+1)}{2}$$

we show $1+2+\cdots+k+(k+1) = \dfrac{(k+1)(k+2)}{2}$.

Lets use what we have; $1+2+\ldots+k+(k+1) = \dfrac{k(k+1)}{2} + k+1$

by inductive hypothesis.

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

This closes the induction. $\square$

Def: we can define a factorial recursively: For a ~~positive~~ non-negative integer n:

$$n! = \begin{cases} 1 & \text{if } n=0 \\ n(n-1)(n-2)\cdots 1 & \text{if } n \geq 1 \end{cases}$$

Ex: $3! = 3 \cdot 2 \cdot 1 = 6$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 4 \cdot 3! = 4 \cdot 6 = 24$

Ex: Show that for positive integers

$$n! \geq 2^{n-1}$$

Pf By Induction: Base case: $n=1$ $\quad 1 \geq 2^0 = 1$ ✓

IH: We assume for some $n=k\geq 1$ $\quad n! \geq 2^{k-1}$

we prove $(n+1)! \geq 2^{n+1-1} = 2^n$

$$(n+1)! = (n+1)\,n! \overset{\text{(by IH)}}{\geq} (n+1)\,2^{n-1} \overset{(n+1)\geq 2}{\geq} 2 \cdot 2^{n-1} = 2^n$$

This closes the induction.

$\square$

More complicated example

<u>Ex</u>: Show that $5^n - 1$ is divisible by 4 for all $n \geq 1$.

<u>Pf</u>: By Induction. <u>Base case</u>: $n=1$ $\quad 5^1 - 1 = 4 \mid 4$ ✓

<u>IH</u>: Assume $5^n - 1$ is divisible by 4 for some $k \geq 1$. We show $5^{n+1} - 1$ is divisible by 4.

$$5^{n+1} - 1 = (5 \cdot 5^n) - 1 = 5 \cdot 5^n \underbrace{-5+4}_{-1}$$

$$= 5(5^n - 1) + 4$$

$$\overset{IH}{=} 5(4 \cdot p) + 4$$

$$= 4m + 4$$

$$= 4t$$

So $5^{n+1} - 1$ is divisible by 4. This closes the induction. $\square$

So far every induction we've looked at so far is an example of <u>weak</u> induction. Weak induction is of the form $P(k) \to P(k+1)$.

However sometimes this is not enough. Sometimes we need more than just the previous case, we need <u>all</u> previous cases.

Strong induction is of the form $\left(P(1) \wedge P(2) \wedge \cdots \wedge P(k)\right) \to P(k+1)$

The proofs work, mostly, the same.

Ex: Consider the sequence $c_1, c_2, \dots$ defined by

$c_1 = 0 \qquad c_n = c_{\lfloor \frac{n}{2} \rfloor} + n$ for all $n > 1$

for example, $c_2 = c_1 + 2 = 2$

$c_3 = c_1 + 3 = 3$

$c_4 = c_2 + 4 = 6$

Prove $c_n < 2n$ for all $n \geq 1$.

Base case $n = 1$. $c_1 = 0 < 2$ ✓

IH: Assume for all k $1 \leq k < n$ $c_k < 2k$

we prove $c_n < 2n$ $n > 1$

$$
\begin{aligned}
c_n &= c_{\lfloor \frac{n}{2} \rfloor} + n \\
&\overset{IH}{<} 2\lfloor \tfrac{n}{2} \rfloor + n \\
&= n + n \\
&= 2n
\end{aligned}
$$

Thus $c_n < 2n$ for all $n \geq 1$. This closes the induction.

□.

Note: The necessity of Strong induction here. The value of c_n did not depend on c_{n-1}, but some value less than n. Thus we needed all values to be true.

(In this case we only need a particular value, but it was easier to assume them all).

Next time, lots of examples.

Bring your own Examples, Ones I prepare don't show actual process!