The growth of functions

Def: Let f & g be function from $\mathbb{Z} \to \mathbb{R}$ (or $\mathbb{R} \to \mathbb{R}$). We say $f \in O(g)$ if $\exists$ C, k constants s.t.

$$|f(x)| \leq C \cdot |g(x)|$$

$\forall x > k$.

This is read as f is big-oh of g.

Intuitively this is saying f grows slower than g

$\exists k$ - means there is some k that after that f grows slower than g.

$\exists C$ - to get around possible differences in size.

Ex: $f(x) = x$
$g(x) = x^2 + 7$

then $\forall x > 0$ $|f(x)| \leq |g(x)|$ so $f \in O(x^2+7)$

Remember this is about growth!

Ex: $f(x) = 100x + 1000$
$g(x) = x^2$.

Well x^2 grows faster but is smaller than f for a while

to capture this we can say $k=1$ $C = 2000$

$\forall x > 1$ $|100x + 1000| \leq 2000|x^2|$. $f \in O(x^2)$.

Ex: $x^2 \notin O(x)$

Pf by Contradiction. Suppose $\exists C, k$ s.t. $x^2 \leq C \cdot x \quad \forall x > k$.

AWLOG $k > 0 \Rightarrow x > 0$ So $x \leq C$ which means x is bounded.

This contradicts that our inequality holds for all $x > k$.

Thus $x^2 \notin O(x)$.

$\square$

Theorem: Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 \quad a_i \in \mathbb{R}$

Then $f \in O(x^n)$.

Pf: $|f(x)| = |a_n x^n + \dots + a_0|$ Assume $x > \overset{k}{1}$

$\leq |a_n| x^n + \dots + |a_1| x + |a_0|$ (triangle inequality).

$\leq x^n \underbrace{(|a_n| + \dots + |a_0|)}_{C}$

$\square$.

Questions: 1. Is $f_1(n) = 2.5n + 12 \in O(n^2)$? Yes

2. Is $f_2(n) = 2.5n^2 + 10 \in O(n^2)$? Yes

3. Is $f_3(n) = 2.5n^{2.5} + 10n^{1.5} - 132 \in O(n^2)$ No.

4. Describe all functions in $O(1)$. All constants, functions bounded.

$\sin(n)$ $\frac{1}{n}$ etc.

Ex: Is $2^{1.5n} \in O(2^n)$?

This is a bit more complicated: Can we find k, C s.t. $2^{1.5n} \leq k 2^n$ $\forall n \geq C$? For simplicity call $m = 2^n$ then our question becomes:

Is $m^{1.5} \in O(m)$ which we know to be false.

Big-Omega: There are other notations besides Big Oh.

Def: We say $f \in \Omega(g)$ iff $\exists\ C > 0$ & $k > 0$ constants such that
$|f(n)| \geq C|g(n)|$ $\forall n \geq k$

That is eventually f is larger than g.

Ex: 1. $n \in \Omega(n)$

choose $C = k = 1$ $n \geq n$ $\forall n \geq 1$

2. $n \in \Omega(\sqrt{n})$

Choose $k = 2$, $C = 1$ then $n \geq \sqrt{n}$ $\forall n \geq 2$.

3. Describe all functions in $\Omega(n)$.

All functions which grow faster than linearly.

Def: If $f \in O(g)$ & $f \in \Omega(g)$ we say $f \in \Theta(g)$.

This is most precise.