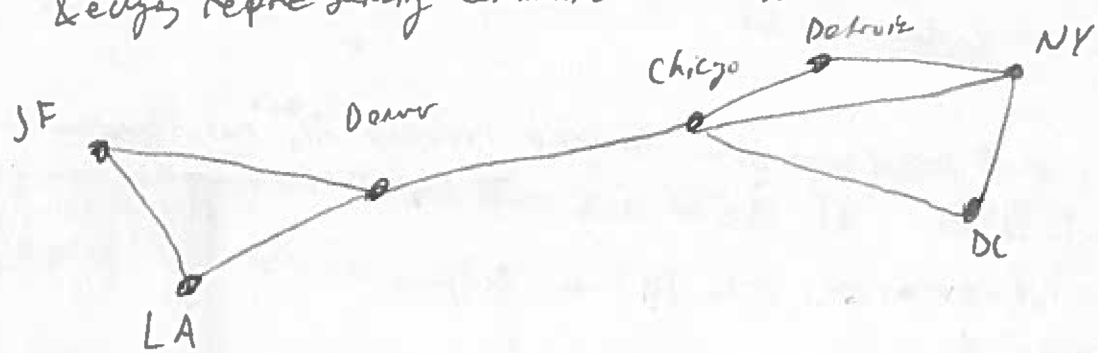
Graphs

Def: A graph $G = (V, E)$ consists of V, a non-empty set of vertices (or nodes) and E, a set of edges. Each edge has either one or two vertices associated with it, called endpoints. An edge connects end points.

Ex: while at their core graphs are just sets we can represent them pictorially. generally we put nodes as dots and edges as lines connecting dots, e.g. we could have nodes representing data centers & edges representing communication between the centers.



Note: for any of the CA cities to talk to the east they must communicate via Denver.

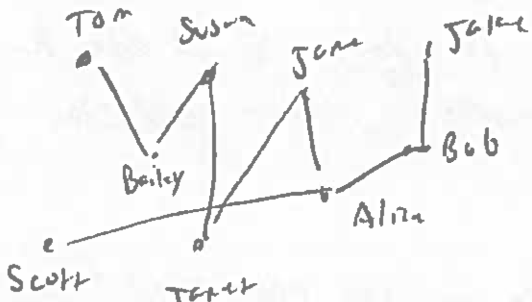
This is a simple graph, graphs can get more complicated if they allow multiple edges between nodes, or self-loops.

Additionally sometimes direction is important & instead of just a line connecting edges we have arrows telling us which way the relationship moves.

Def: A directed graph (digraph) (V, E) consists of nonempty V, vertices & a set of directed edges E. Each directed edge is an ordered pair of vertices. (u, v) is said to start at u & end at v.

Ex: Graphs are often convenient ways of displaying information.

Such as who are friends on Social Media:

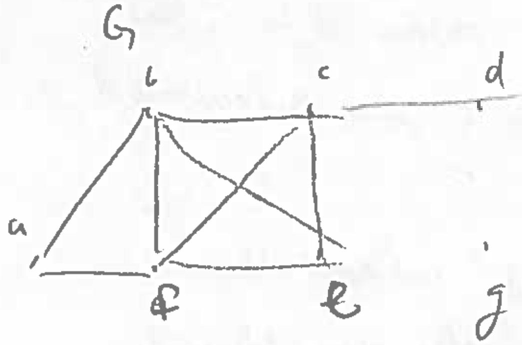


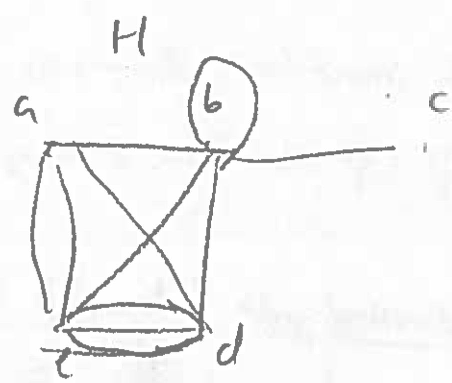
Def: Two vertices u, & v in an undirected graph are called adjacent (or neighbors) in G if u & v are endpoints of an edge e of G. e is said to connect u & v.

Def: The set of all neighbors of a vertex v is called the neighborhood of v denoted N(v). If $A \subseteq V$ we call N(A) the neighborhood of A, the set of all vertices in G that are adjacent to at least one of the vertices in A.

Def: The degree of a vertex in an undirected graph is the number of edges connected to it. Except that a loop contributes twice to the degree of that vertex, degree is denoted deg(v).

Ex: Given





What are the degrees of each node & the neighborhoods of each node?

G: $\deg(a) = 2$ $\quad$ $\deg(b) = 4$

$N(a) = \{f, b\}$ $\quad$ $N(b) = \{a, f, e, c\}$

$\deg(c) = 4$ $\quad$ $\deg(d) = 1$ $\quad$ $\deg(e) = 3$

$N(c) = \{b, f, e, d\}$ $\quad$ $N(f) = \{c\}$ $\quad$ $N(e) = \{f, b, c\}$

$\deg(f) = 4$ $\quad$ $\deg(g) = 0$

$N(f) = \{a, b, c, e\}$ $\quad$ $N(g) = \{\}$

H: $\deg(a) = 4$ $\quad$ $\deg(b) = 6$

$N(a) = \{e, d, b\}$ $\quad$ $N(b) = \{a, b, c, d, e\}$

$\deg(c) = 1$ $\quad$ $\deg(d) = 5$ $\quad$ $\deg(e) = 6$

$N(c) = \{b\}$ $\quad$ $N(d) = \{e, a, b\}$ $\quad$ $N(e) = \{a, b, d\}$

Theorem (Hand Shaking Thm): Let G be a graph with m undirected edges

Then $2m = \sum_{v \in V} \deg(v)$

The deg(v) counts number of edges on each node, so each edge gets double counted.

Ex: How many edges are there in a graph with 10 nodes each of degree 6?

By Thm Sum of degrees is $60 \Rightarrow 30$ total edges

Theorem: An undirected graph has an even number of vertices of odd degree

pf: Let V_1 = nodes of even deg $\quad$ V_2 = nodes of odd deg.

$$2m = \sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v_2)$$

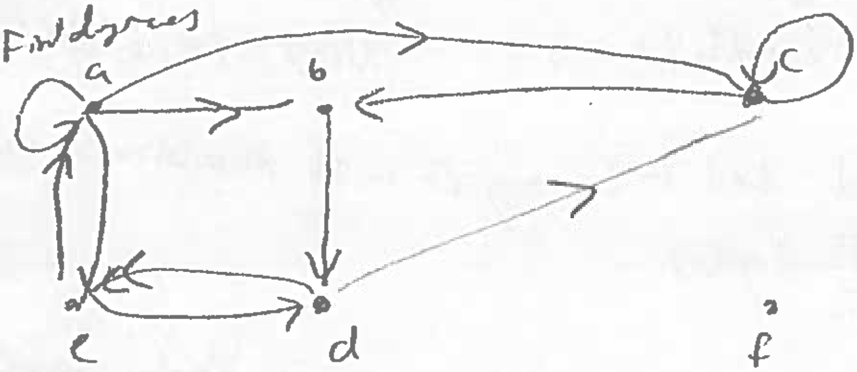
$\deg(v)$ is even for each $v \in V_1$ both $2m$ & $\sum_{v \in V_1} \deg(v)$ are even

Thus $\sum_{v \in V_2} \deg(v)$ must also be even but each term is odd $\Rightarrow$ even number of odd terms!. $\square$

Def: When (u,v) is an edge of the directed graph G. u is said to be adjacent to v. v is adjacent from u. u is called the initial vertex & v the terminal vertex.

Def: In a directed graph the in-degree of a vertex v, denoted $\deg^-(v)$ is the number of edges with v as their terminal vertex. The out-degree of v is the number of edges with v as the initial vertex, denoted $\deg^+(v)$

Ex: Find degrees



$\deg^-(a) = 2$

$\deg^+(a) = 4$

$\deg^-(d) = 2$

$\deg^+(d) = 2$

$\deg^-(b) = 2$

$\deg^+(b) = 1$

$\deg^-(e) = 2$

$\deg^+(e) = 2$

$\deg^-(c) = 3$

$\deg^+(c) = 2$

$\deg^-(f) = 0$

$\deg^+(f) = 0$

Theorem: Let $G = (V,E)$ be a directed graph. Then

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|.$$