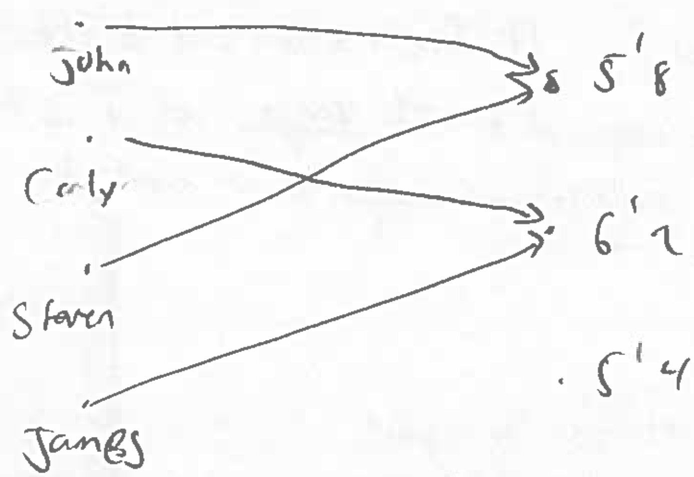
Functions

In Many instnces we wish to associate elements in one set to elements in another (or the Same set).

For example we might have a set of ppl {John, Carly, Steven, James} & heights: {5'8, 6'2, 5'4}

And we might want to associate ppl w/ their height



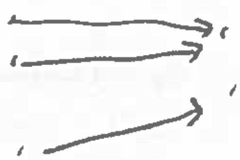
Def: Let A & B be non-empty sets. A <u>function</u> f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a)=b$ for b being the <u>unique</u> element of B assigned by f to a in A. We write $f: A \to B$

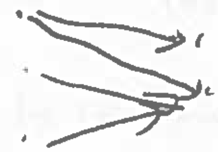
Note: We may specify functions in various ways, explicitly $f(x)=x+1$ or implicitly, $f: A \to B \subseteq A \times B$

elements are (a,b) meaning $f(a)=b$.

Note: The assignment of a to b by f is the only assignment of a by f allowed.

e.g. function ✓ Not a function ✗





Given $a \in A$ there must be a unique $b \in B$ s.t. $f(a) = b$.

Def: If $f: A \to B$ is a function we call A the domain of f & B is the co-domain of f. If $f(a) = b$ we call b the image of a & a a pre-image of b. The range of f is the subset of B which is the collection of images of all $a \in A$. We sometimes say f maps A to B.

Note: range$(f) \subseteq B$ not necessarily equal

Ex: $f: \mathbb{R} \to \mathbb{Z}$ $\quad f(x) = |\lfloor x \rfloor|$

$\mathbb{R}$ = domain $\quad$ range = $\mathbb{N}$

$\mathbb{Z}$ = co-domain

Two functions are equal if they have the same domain, co-domain & map each element of the domain to the same element of the codomain.

Ex: $f: \{(1,a), (2,a), (3,b)\}$

is NOT a fcn of $X = \{1,2,3,4\}$ to $Y = \{a,b,c\}$ b/c no mapping for $4 \in X$

Is a fcn from $A = \{1,2,3\}$ to Y even though c is not mapped to.

range of $f = \{a,b\}$.

<u>Def</u>: The <u>preimage</u> of an element of the co-domain is a subset of the domain where each element maps to the initial given element.

$pre_f(b) = \{x ; x \in A \ \& f(x) = b\}$

<u>Ex</u> $f : \{(1,a), (2,a), (3,b)\}$

$\{1,2,3\} \to \{a,b,c\}$

$pre_f(a) = \{1,2\}$ $\quad pre_f(b) = \{3\}$ $\quad pre_f(c) = \emptyset$

$pre_f(2)$ non sense.

<u>Ex</u>: Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.

domain : $\mathbb{R}$ $\quad$ range : $\mathbb{R}^{\geq 0}$

co-domain: $\mathbb{R}$

$pre_f(5) = \{\sqrt{5}, -\sqrt{5}\}$ $\quad pre_f(0) = \{0\}$

$pre_f(-5) = \emptyset$.

<u>Def</u>: Let f, g be functions from A to $\mathbb{R}$ then $f + g$ & fg are also functions from A to $\mathbb{R}$

$\forall x \in A$ $(f+g)(x) = f(x) + g(x)$

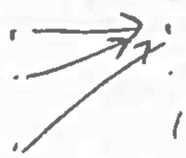
$(fg)(x) = f(x) \cdot g(x)$.

Ex. If $f(x) = x^2$ $g(x) = 5-x$ what are $f+g$ & fg?

$f+g\,(x) = x^2 + 5 - x$ $fg\,(x) = x^2(5-x) = 5x^2 - x^3$

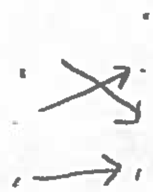
Def: A function f is said to be one-to-one or injective iff $f(a) = f(b)$ implies $a = b$, or if $a \neq b$ the $f(a) \neq f(b)$.

Ex For a general function



this is fine. Injective means each element of the co-domain is mapped to at most once.

not injective.



← injective

Ex: $f(x) = \lfloor x \rfloor$ $\mathbb{R} \to \mathbb{Z}$ is NOT 1-to-1.

$f(5.4) = f(5.3)$ but $5.4 \neq 5.3$.

Ex: prove $f(x) = 5x + 7$ is one-to-one.

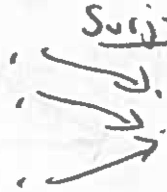
Pf: Choose $x \neq y$ then $5x \neq 5y \Rightarrow 5x + 7 \neq 5y + 7$.

$\blacksquare$

Def: A function f from A to B is called Onto or Surjective iff for every element b of the Codomain $\exists\, a \in A$ s.t. $f(a) = b$.

e.g. Not Surjective

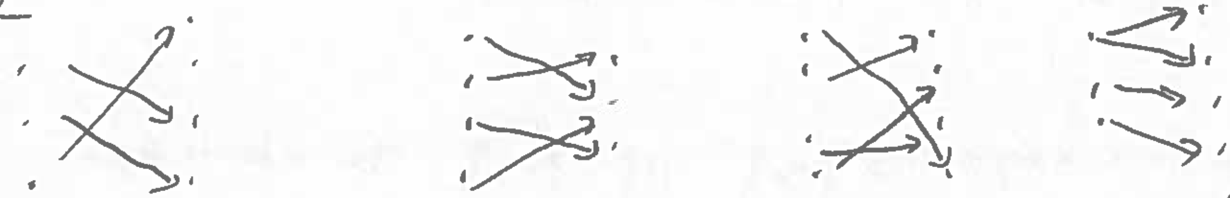
Surjective not injective.



Ex: $f(x) = x^2$ $\mathbb{Z} \to \mathbb{Z}$ is not surjective no x maps to -3

$\mathbb{Z} \to \mathbb{N}$ is not either no x maps to 3

Ex



fcn, one-to-one   fcn, surjective   fcn, surj, inj.   not a fcn,

Def: The function f is a bijection if it is both injective & surjective.

Note domain & co-domain play a big part in being inj & surj.

A function is ALWAYS surjective onto its image (range) by definition, it's the set of things mapped to.

Def: If a fcn f is a bijection from $A \to B$ then $\exists$ an inverse fcn $f^{-1}: B \to A$ s.t. $f(f^{-1})$ is the identity fcn. $f(f^{-1})(b) = b$ & $f^{-1}(f)(a) = a$.

Don't mistake f^{-1} for $\frac{1}{f}$ the -1 is not a power in this case.

Ex: $f(x) = x^2$ $\mathbb{Z} \to \mathbb{Z}$ is not invertible (not 1-to-1)

$\mathbb{N} \to \mathbb{N}$ is not (not onto).

$\mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is $f^{-1} = \sqrt{x}$

More exs on Hmwk. ☺