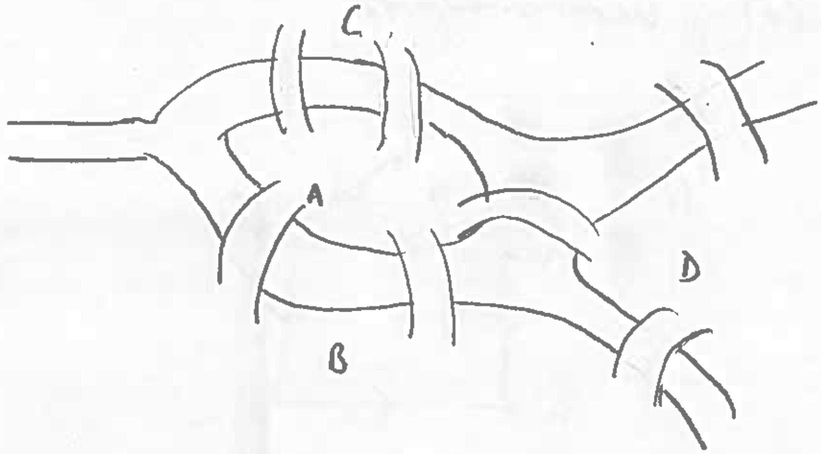
Euler and Hamilton Paths

Leading question: Can we travel our whole graph (every edge) and return to our starting vertex? Can we do it while only travelling each edge exactly once?

What about visiting each vertex exactly once?

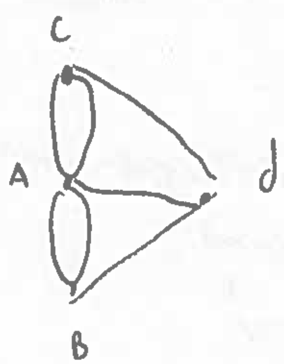
The first is an Euler circuit, the second a Hamiltonian circuit.

The original problem came from the town Königsberg, Prussia was divided by multiple rivers as such with the following bridges



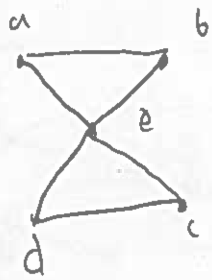
The question was Could you travel each bridge (seeing the sights) exactly once?

Euler solved this in 1736 using (creating) graph theory. The associated graph is

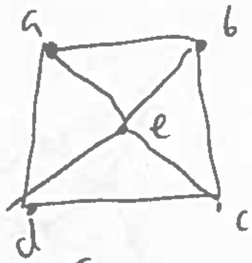


__Def__: An __Euler circuit__ in a graph G is a simple circuit containing every edge in G. An __Euler path__ in G is a simple path containing every edge of G.

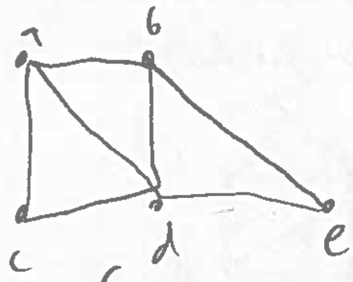
Ex: Which of the following have Euler circuit? If not does it have an Euler path?



G_1



G_2



G_3

G_1 has Euler circuits: G_1, a, b, c, d, e, a

G_2 does not, & has no Euler path either

G_3 has an Euler path, but no circuit a, b, d, c, a, e, b.

How to determine this in general? Let's examine a fictional graph & count deg of nodes. Assuming we start with a & move to b $\Rightarrow \deg(a) \geq 1 \quad \deg(b) \geq 1$ since (a,b) is an edge. To be a circuit we must leave b via another edge $\Rightarrow \deg(b) \geq 2$. In particular any node we visit we must also leave via a new edge $\Rightarrow \deg(v)$ is even! Note we might revisit a adding some even degree, but eventually we must end at a too $\Rightarrow \deg(a) = 1 + \text{even} + 1 = \text{even}$.

Thus in order to have an Euler circuit every vertex must have even degree.

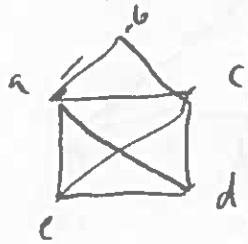
This is a necessary condition for Euler circuits. Is it sufficient? Meaning, do all graphs of all even degrees have a Euler cycle?

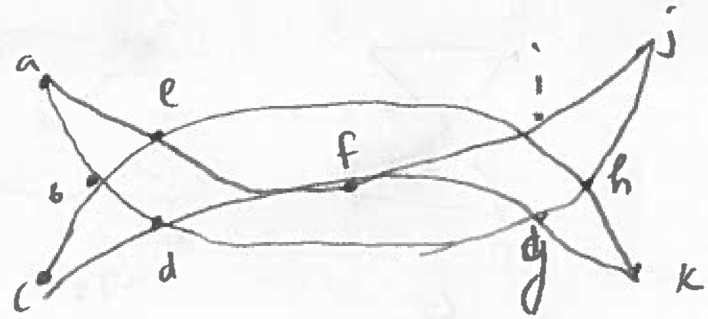
The answer is Yes! the construction is in the book.

<u>Theorem</u>: A connected multigraph with at least two vertices has an Euler circuit iff each vertex has even degree.

Ex: Many puzzles ask you to draw them without lifting your pencil

Can you do it?





e, ~~c~~, a, b, c, d, a, e

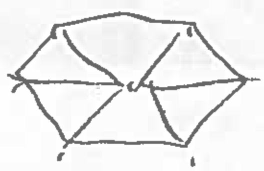
a, b, d, g, h, j, i, h, k, g, f, d, c, b, e, i, f, e, a

There is an algorithm in the book for finding such circuits.

Theorem: A connected multigraph has an Euler path (and not Euler circuit) if and only if it has exactly two vertices of odd degree.

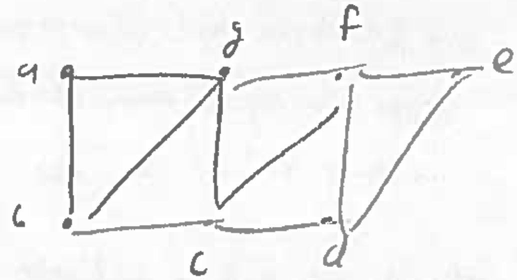
Those vertices are your starting & ending vertices.

Ex:



No Euler path b/c all edge nodes have odd degree.

Ex:



has an Euler path only b, d have odd deg

b, a, g, b, c, g, f, c, d, f, e, d.