Equivalence relations:

<u>Def</u>: a relation R on A is an <u>equivalence relation</u> if it is symmetric, reflexive, & transitive.

Equivalence relations often give nice ways of saying two things are equivalent (the same) when they are different.

<u>Ex</u>: $A = \mathbb{Z}$, $R = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : 4 \mid a-b\}$ or a, b congruent mod 4.

$(0,4) \in R$, $(1,9) \in R$

note $\forall a \in \mathbb{Z}$ $(a,a) \in R$ ($a-a=0$ $4 \mid 0$ ✓) $\Rightarrow$ reflexive

If $(a,b) \in R \Rightarrow a-b = 4k \Rightarrow b-a = -4k \Rightarrow (b,a) \in R$

$\Rightarrow$ Symmetric

If $(a,b) \in R$ & $(b,c) \in R$ then $a-b = 4k$ $b-c = 4j$

$a-(4j+c) = 4k$

$\Rightarrow a-c = 4k+4j$

$\Rightarrow (a,c) \in R \Rightarrow$ transitive.

modulo congruence is an equivalence relation, we know it gives a nice way of saying two different numbers are the same,

<u>Ex</u>: Let A = all strings $R = \{(a,b) \in A \times A : len(a) = len(b)\}$

Is R an equivalence relation?

Yes. This is saying if all we judge strings by is their length then all strings of the same length are the same! (equivalent).

Def: Let R be an equivalence relation on A. The set of all elements that are related to an element $a \in A$ is called the equivalence class of a.
This is denoted $[a]$

That is $[a] = \{ s \in A : (a,s) \in R \}$

$b \in [a]$ is called a representative of the equivalence class.

Ex: What is $[0]$ when our relation is congruence modulo 4?

$[0] = \{ \ldots -8, -4, 0, 4, 8, \ldots \}$ all multiples of 4

Ex: In C you can name your variables anything you want, however some older compilers only checked the first 8 characters of a variable.

Thus [Number_of_tropical_storms] = all strings of the form Number_of ← wildcard.

These variables were considered the same.

Theorem: Let R be an equivalence relation on A. The following are equivalent

(i) $a R b$
(ii) $[a] = [b]$
(iii) $[a] \cap [b] \neq \emptyset$.

Recall, TFAE proofs mean that all claims are saying the same thing & we need to show they imply each other.

__Pf__: (i) $\Rightarrow$ (ii) If aRb then $[a] = [b]$

Choose $c \in [a]$ then aRc by assumption aRb & R reflexive $\Rightarrow bRa$

$\Rightarrow bRa$ & aRc & R transitive $\Rightarrow bRc \Rightarrow c \in [b]$

the reverse is the same. Thus $[a] \subseteq [b]$ & $[b] \subseteq [a] \Rightarrow [a] = [b]$.

(ii) $\Rightarrow$ (iii) $[a] = [b] \Rightarrow [a] \cap [b] \neq \emptyset$

$[a]$ is non empty $a \in [a] \Rightarrow a \in [b] \Rightarrow a \in [a] \cap [b]$

So $[a] \cap [b] \neq \emptyset$.

(iii) $\Rightarrow$ (i) $[a] \cap [b] \neq \emptyset \Rightarrow aRb$

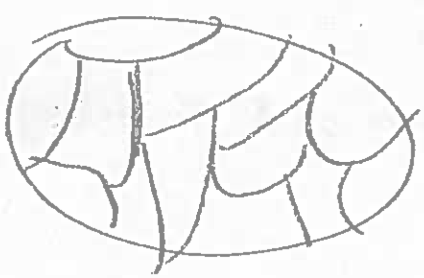
$\exists c \in [a] \cap [b] \Rightarrow cRa$ & cRb R reflexive

$\Rightarrow aRc \Rightarrow aR\cancel{c}$ & cRb & R transitive

$\Rightarrow aRb$.

$\square$

__Def__: a __partition__ of a set S is a collection of disjoint non-empty subsets whose union is S i.e. splitting S into distinct parts.



__Theorem__: Let R be an equivalence relation on A. The equivalence classes of R partition A.

Congruence classes are a great example of this