Binomial Theorem

Def: A binomial is a sum of two terms $x+y$

Ex: We can compute $(x+y)^3$ in a combinatorial method rather than actually cubing. We might know all our terms are x^3, x^2y, xy^2, y^3

the question is how many ways do each show up?

$(x+y)^3 = (x+y)(x+y)(x+y)$

x^3 - shows up if we multiply all x's
$\Rightarrow$ 1 way

x^2y - shows up if we choose 2 x's & 1 y
$\Rightarrow \binom{3}{2}$ ways

$xy^2 \Rightarrow x$ is chosen 1 time
$\Rightarrow \binom{3}{1}$ ways

$y^3 \Rightarrow$ 1 way

$$\Rightarrow (x+y)^3 = x^3 + \binom{3}{2}x^2y + \binom{3}{1}xy^2 + y^3$$

Binomial Theorem: Let x, y be variable $n \in \mathbb{N}$

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \dots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

Proposition: $\binom{n}{r} = \binom{n}{n-r}$

Pf: $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!(n-(n-r))!} = \binom{n}{n-r}$

$\square$

Pf 2: $\binom{n}{r}$ counts the ways to choose r elements from n elements, identically we can count the elements which are not selected $n-r$ are not selected $\Rightarrow \binom{n}{n-r} = \binom{n}{r}$.

$\square$

Ex: Expand $(x+y)^4$

$$(x+y)^4 = \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j$$

$$= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4$$

$$= x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4$$

Ex: What is the Coefficient infront of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$

Binom theorem $\Rightarrow$

$$(2x-3y)^{25} = \sum_{j=0}^{n} \binom{25}{j} (2x)^{25-j} \cdot (-3y)^j$$

$x^{12}y^{13}$ occurs when $j=13$

$\binom{25}{13} (2x)^{12} (-3y)^{13} \Rightarrow \binom{25}{13} \cdot 2^{12} \cdot (-3)^{13}$ is our coefficient.

Proposition: Let $n \in \mathbb{N}$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Pf: $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k}$. $\square$

or Pf: A set with n elements has 2^n subsets. The subsets are of size $0, 1, 2, 3, \dots, n$
there are $\binom{n}{0}$ subsets of size 0 $\binom{n}{1}$ subsets of size 1 etc

$$\Rightarrow \sum_{k=0}^{n} \binom{n}{k} \text{ total subsets } \Rightarrow 2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

Pascals Identity:

Theorem: (pascals identity) Let $n, k \in \mathbb{N}$, $n \geq k$

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Pf: Let T be a set $|T| = n+1$. $a \in T$, $S = T \setminus \{a\}$
we know there are $\binom{n+1}{k}$ subsets of T containing k elements.
A given subset with k elements either contains a or doesn't.
Meaning we can count subsets of T with k elements another way.

Case 1: Contains a $\Rightarrow \binom{n}{k-1}$ ways to do this

Case 2: Does not contain a $\Rightarrow \binom{n}{k}$ ways to do this

Case 1 & 2 are mutually exclusive $\Rightarrow$ There are $\binom{n}{k-1} + \binom{n}{k}$
subsets of T containing k elements. $\square$

Significance? Pascal's triangle!

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\vdots$$

$$\downarrow$$

$$\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$$

$$1$$

$$1 \quad 1$$

$$1 \quad 2 \quad 1$$

$$1 \quad 3 \quad 3 \quad 1$$

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

$$\vdots$$

rows look familiar?