

Binomial Theorem

Def: A binomial is a sum of two terms $x+y$

Ex: We can compute $(x+y)^3$ in a combinatorial method rather than actually cubing. We might know all our terms are x^3, x^2y, xy^2, y^3

the question is how many ways do each show up?

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

x^3 - shows up if we multiply all x 's

$\Rightarrow 1$ way

x^2y - shows up if we choose 2 x 's & 1 y

$\Rightarrow \binom{3}{2}$ ways

$xy^2 \Rightarrow x$ is chosen 1 time

$\Rightarrow \binom{3}{1}$ ways

$y^3 \Rightarrow 1$ way.

$$\Rightarrow (x+y)^3 = x^3 + \binom{3}{2} x^2y + \binom{3}{1} xy^2 + y^3$$

Binomial Theorem: Let x, y be variable $n \in \mathbb{N}$

$$(x+y)^n = \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \dots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

Proposition: $\binom{n}{r} = \binom{n}{n-r}$

Pf: $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!(n-(n-r))!} = \binom{n}{n-r}$ \square

Pf 2: $\binom{n}{r}$ counts the ways to choose r elements from n elements, identically we can count the elements which are not selected $n-r$ are not selected $\Rightarrow \binom{n}{n-r} = \binom{n}{r}$. \square

Ex: Expand $(x+y)^4$

$$(x+y)^4 = \sum_{j=0}^4 \binom{4}{j} x^{4-j} y^j$$

$$= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4$$
$$= x^4 + 4x^3 y + 6x^2 y^2 + 4x y^3 + y^4$$

Ex: What is the coefficient of $x^{12} y^{13}$ in the expansion of $(2x-3y)^{25}$

Binom theorem \Rightarrow

$$(2x-3y)^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} \cdot (-3y)^j$$

$x^{12} y^{13}$ occurs when $j=13$

$$\binom{25}{13} (2x)^{12} (-3y)^{13} \Rightarrow \binom{25}{13} \cdot 2^{12} \cdot (-3)^{13} \text{ is our coefficient.}$$

Proposition: Let $n \in \mathbb{N}$

$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

Pf: $2^n = (1+1)^n = \sum_{k=0}^n \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^n \binom{n}{k}$

□

or pf: A set with n elements has 2^n subsets. The subsets are of size $0, 1, 2, 3, \dots, n$
there are $\binom{n}{0}$ subsets of size 0 $\binom{n}{1}$ subsets of size 1 etc

$$\Rightarrow \sum_{k=0}^n \binom{n}{k} \text{ total subsets} \Rightarrow 2^n = \sum_{k=0}^n \binom{n}{k}.$$

Pascal's Identity:

Theorem: (Pascal's Identity) Let $n, k \in \mathbb{N}$, $n \geq k$

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

pf: Let T be a set $|T| = n+1$. $a \in T$, $S = T \setminus \{a\}$
we know there are $\binom{n+1}{k}$ subsets of T containing k elements.

A given subset with k elements either contains a or doesn't.
Meaning we can count subsets of T with k elements another way.

Case 1: Contains $a \Rightarrow \binom{n}{k-1}$ ways to do this

Case 2: Does not contain $a \Rightarrow \binom{n}{k}$ ways to do this

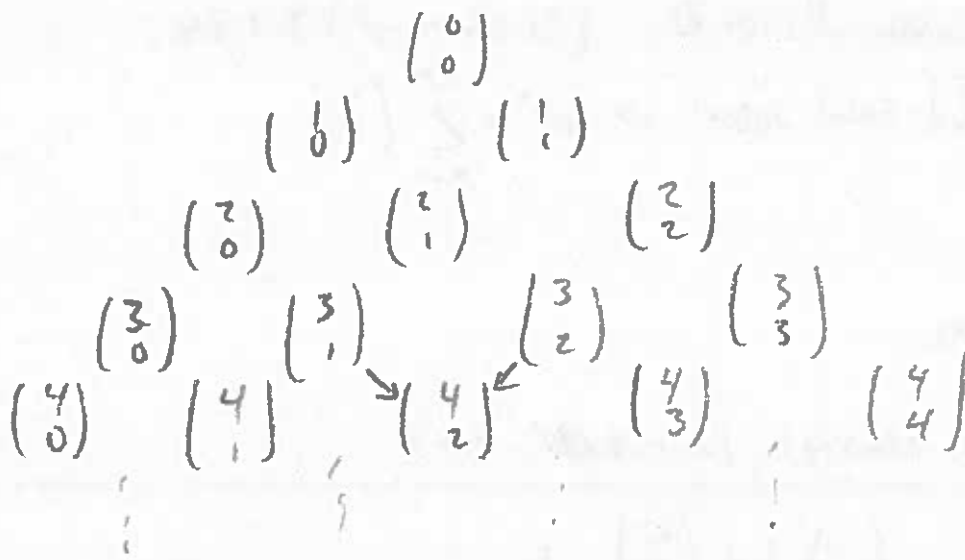
Case 1 & 2 are mutually exclusive \Rightarrow there are $\binom{n}{k-1} + \binom{n}{k}$

subsets of T containing k elements.

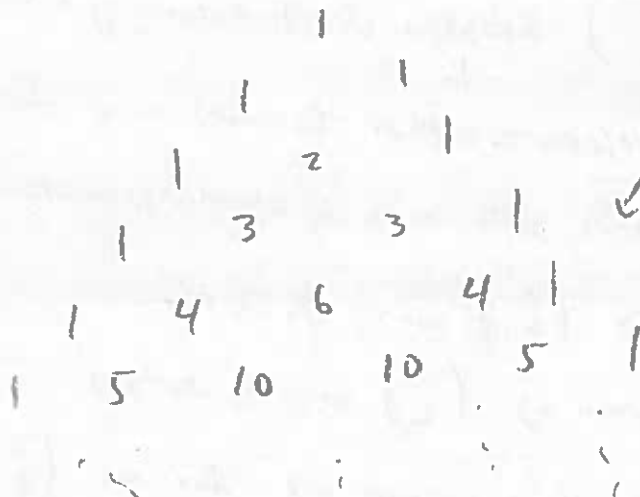
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Significance?

Pascal's triangle!



$$\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$$



rows look familiar?