CSCI 2824 - Discrete Structures
Homework 6

You MUST show your work. If you only present answers you will receive minimal credit. This homework is worth 100pts.

Due: Wednesday July 19

1. For the following question consider rolling two dice. One dice is red and one is blue, but they are otherwise normal dice.

   (a) (4 points) How many outcomes result in a sum of 2? In a sum of 12?

   Solution: Only one outcome results in a sum of 2, 1 and 1.
   Only one outcome results in a sum of 12, 6 and 6.

   (b) (4 points) How many outcomes have the blue die showing 2?

   Solution: Six outcomes have the blue die showing 2. Any roll with the red die (of which there are 6).

   (c) (4 points) How many outcomes give an even sum?

   Solution: Possible even sums are 2, 4, 6, 8, 10, 12.
   There is only one way to make 2.
   There are 3 ways to make 4, 2 and 2, red 3 blue 1 and red 1 blue 3.
   There are 5 ways to make 6, 3 and 3, red 4 blue 2, red 2 blue 4, red 5 blue 1, and red 1 blue 5.
   There are 5 ways to make 8, 4 and 4, red 5 blue 3, red 3 blue 5, red 6 blue 2, and red 2 blue 6.
   There are 3 ways to make 10, 5 and 5, red 6 blue 4, and red 4 blue 6.
   There is only one way to make 12.
   Altogether there are 18 ways to get an even sum.

2. For the following questions consider an eight-bit string.

   (a) (2 points) How many eight-bit strings begin with 1100?

   Solution: We know the first four bits of the eight-bit string, the last 4 have 2 choices each. There are $2^4 = 16$ eight-bit strings with 1100 in the first four bits.

   (b) (2 points) How many eight-bit strings have exactly one 1 in them?

   Solution: If there is exactly one 1 then the other 7 bits are all 0. Meaning we only have to choose which place to put the 1 in. There are $\binom{8}{1} = 8$ ways to place the 1.

   (c) (5 points) How many eight-bit strings have exactly two 1's in them?

   Solution: We know there will be two 1s and six 0s the only question is where to place the two 1s. There are $\binom{8}{2} = 28$ ways to place the two 1s.

   (d) (5 points) How many eight-bit strings have at least one 1 in them?

Solution: The long way to do this problem is to compute how many strings have one 1, how many have two 1*s* etc. The shorter is to compute the opposite. How many have no 1s. There is only one eight-bit string which has no 1s, every other string has at least one.

Thus there are $2^8 - 1 = 255$ eight-bit string which have at least one 1 in them.

3. For the following questions determine how many permutations (orderings) can be formed from $\{A, B, C, D, E\}$ subject to the given constraints.

   (a) (4 points) The ordering contains the substring ACE (in that order directly, as in $ACEDB$ is one, but $ADCBE$ is not)

   Solution: First we compute how many places can ACE go. It could start at the first, second, or third character. Meaning there are three ways of choosing where to place ACE.

   There are 2 ways of ordering the remaining two letters.

   Thus altogether we have $3 \cdot 2 = 6$ ways of forming strings with ACE as a substring.

   (b) (6 points) Does not contain the substring AB nor CD (again directly, $ACEDB$ is acceptable).

   Solution: To compute this we work with the opposite. That is we compute how many strings have AB, have CD and have AB AND CD and then subtract that from the total number of strings. Note we have to compute the strings that have both AB and CD and not do it separately so that we don't double count some strings.

   To count the number of strings that contain just AB is just 4! since we have to permute 4 objects: AB, C, D, E.

   Similarly to count the number of strings that contain just CD is 4! as well.

   To count the strings that contain both AB and CD is 3! since we are permuting 3 object, AB, CD, E.

   There are $5! = 120$ total strings from $\{A, B, C, D, E\}$ so there are $120 - (24 + 24 - 6) = 78$ strings which do not contain AB nor CD as substrings. We need to subtract the 6, is because adding the two separate cases double counts the cases where they both show up.

   (c) (10 points) A appears before C which appears before E (here there can be letters between).

   Solution: For this problem we will recognize why part of the total strings of length 5 are acceptable. There are $5! = 120$ possible strings. The possible cases (which are all equally likely and have the same number of cases are):

   - A appears before C which appears before E (acceptable).
   - A appears before C but E appears before C (not acceptable).
   - C appears before A which appears before E (not acceptable).
   - C appears before E which appears before A (not acceptable).
   - E appears before A which appears before C (not acceptable).
   - E appears before C which appears before A (not acceptable).

Each of these six options have just as many string which obey their rules. And all strings obey one of the rules above. Meaning that only a sixth of the total number of strings is acceptable. So there are

$$\frac{5!}{6} = 20$$

strings which have A before C which is before E.

4. (10 points) Prove that a number is divisible by 3 if and only if the sum of its digits is divisible by 3. (E.g. 27 is divisible by 3 since $2 + 7 = 9$ is divisible by 3).

Solution: ($\Rightarrow$) If a number, n, is divisible by 3 then $n = 3k$ for some integer k. We can write n as multiples of powers of 10:

$$n = a_k 10^k + a_{k-1} 10^{k-1} + \cdots + a_1 10 + a_0$$

This is simply the base 10 expansion of n.

Thus we have the equation:

$$a_k 10^k + a_{k-1} 10^{k-1} + \cdots + a_1 10 + a_0 = 3k$$

Now take both sides modulo 3, noting that $10 \mod 3 \equiv 1$

$$a_k + a_{k-1} + \cdots + a_1 + a_0 \mod 3 \equiv 0$$

Thus the sum of the digits is also a multiple of 3.

($\Leftarrow$) If the sum of the digits of a number is a multiple of three we can do the above proof backwards, given that the sum is a multiple of 3 multiplying each term by 1 does nothing to affect the sum, 10 is equivalent to 1 modulo 3 thus n is also a multiple of 3.

5. (15 points) All numbers have some divisibility rule associated with them. There are a couple for 7. The one we focus on here is that a number n is divisible by 7 if and only if when you subtract 2 times the least significant digit from the number without its least significant digit the result is also divisible by 7. E.g. 14 is divisible by 7 since $1 - 4 \cdot 2 = -7$ is divisible by 7. 343 is divisible by 7 since $34 - 3 \cdot 2 = 28$ which is divisible by 7. Prove this.

Solution: To prove this we do a similar trick as the previous problem: If a number is divisible by 7 we can write $n = 7k$ for some integer k. Write n in a particular form:

$$n = a_1 10 + a_0$$

E.g. we could write $30248 = 3024 \cdot 10 + 8$

Then our equation becomes:

$$a_1 10 + a_0 = 7k$$

Now take both sides modulo 7, noting that $10 \mod 7 \equiv 3$

$$a_1 3 + a_0 \mod 7 \equiv 0$$

This gives a rule that we could split a number up by taking off the last digit, multiplying the number made up of the large digits by 3 then add the remaining digit, e.g. 343 is divisible by 7 since $34 \cdot 3 + 3 = 105$ and 105 is divisible by 7 since $10 \cdot 3 + 5 = 35$ which is divisible by 7.

This rule however doesn't scale well at all. Imagine needing to check whether 13983, were divisible by 7 in your head, multiplying 1398 by 3 will be a lot of work.

Thus we keep working with our equation above. It would be convenient if we could get rid of the multiple of the higher digits. Thus we need to multiply by the inverse of 3 modulo 7. $3^{-1} \mod 7 \equiv 5$. Thus our rule becomes:

$$a_1 + a_0 5 \mod 7 \equiv 0$$

Which is better, we're only multiplying a small number by 5. However we can actually make this easier by remembering that $5 \equiv -2 \mod 7$ thus our equation is $a_1 - 2a_0 = 7k$.

Thus our algorithm is to split the least significant digit off the number, multiply that by 2 and subtract it from the number without the least significant digit.

6. For the following questions find the number of (unordered) five-card poker hands, selecting from an ordinary 52-card deck, having the properties indicated.

   (a) (5 points) Containing four of a kind, that is, four cards of the same denomination.

   Solution: In order to get a four of a kind we must first choose which of the 13 denominations we will duplicate: there are $\binom{13}{1}$ options for this.
   Then we must choose which of the remaining 12 denominations we will allow for our remaining card: $\binom{12}{1}$ ways to do this.
   Then we must choose which suit it will be, there are $\binom{4}{1}$ ways this could fall.
   Thus there are $\binom{13}{1} \cdot \binom{12}{1} \cdot \binom{4}{1} = 624$

   (b) (5 points) Containing cards of exactly two suits

   Solution:
   This one is complicated we need to first over count and then eliminate some possibilities.
   Initially we count there are $\binom{4}{2}$ ways to count which suits are represented and then there are $\binom{26}{5}$ ways of choosing our five cards from among the 26 possible (only 2 suits allowed remember!). This seems good, BUT this count also includes the possibility that we selected all the same suit. Thus we need to subtract some values.
   Thus we need to count how many hands have cards from one suit, essentially a flush. There are $\binom{4}{1}$ ways of choosing which suit and $\binom{13}{5}$ ways of choosing those cards.
   Actually when we counted the possible hands with two suits we actually counted each of the ways with one suit 3 times (can you see why?)
   Thus there are:

$$\binom{4}{2} \cdot \binom{26}{5} - 3 \cdot \binom{4}{1} \cdot \binom{13}{5} = 379,236$$

   ways of getting exactly 2 suits.

(c) (5 points) Containing two of one denomination, two of another denomination and one of a third denomination.

Solution: In order to choose the first two we first see how many ways we have to choose a denomination: $\binom{13}{1}$.
Then we must select two of this denomination: $\binom{4}{2}$.
Then we must select a denomination for the second two: $\binom{12}{1}$.
And which cards are selected: $\binom{4}{2}$.
Finally we select the third denomination: $\binom{11}{1}$.
And the card of that denomination: $\binom{4}{1}$.
Altogether we have:

$$\binom{13}{1} \cdot \binom{4}{2} \cdot \binom{12}{1} \cdot \binom{4}{2} \cdot \binom{11}{1} \cdot \binom{4}{1} = 247,104$$

ways of forming hands of that type.

7. The following questions deal with selecting a committee from a club consisting of six distinct men and seven distinct women.

(a) (4 points) In how many ways can we select a committee of three men and four women?

Solution: $\binom{6}{3} \cdot \binom{7}{4} = 700$.

(b) (4 points) In how many ways can we select a committee of four persons that has at least one man?

Solution: $\binom{13}{4} = 715$ ways to choose a committee of four people, but we need to eliminate all of the ones with no men which is $\binom{7}{4} = 35$ so there are 680 ways.

(c) (6 points) In how many ways can we select a committee of four persons that has at most one man?

Solution:
From the previous problem we have seen there are 35 ways to select a committee with no man.
Now just count the ways to choose a committee with one man:
$\binom{6}{1} \cdot \binom{7}{3} = 210$ ways. Thus there are 245 ways to select a committee with at most one man.