CSCI 2824 - Discrete Structures
Homework 3

You MUST show your work. If you only present answers you will receive minimal credit. This homework is worth 100pts.

Due: Wednesday June 28

1. (4 points) For each of the following determine the number of elements in the given set.

   (a) $\{\}$

   Solution: 0

   (b) $\{\{\}, \{\{\}\}\}$.

   Solution: 2

   (c) $\{a, b, \{\}, \{\{\{\}\}\}\}$

   Solution: 4

   (d) $\{a, b, \{a, b\}, \{a, c\}, \{a\}\}$

   Solution: 5

2. (5 points) For the following pairs of sets, determine which operator goes between the sets to make a true statement: $\in$, $\ni$, $\subseteq$, $\supseteq$, or none.

   (a) $\{1, 2\}$, $\{1, 2, \{1, 2\}\}$

   Solution: $\{1, 2\} \in \{1, 2, \{1, 2\}\}$. However $\{1, 2\} \subseteq \{1, 2, \{1, 2\}\}$ as well.

   (b) $\{1, 2\}$, $\mathbb{N}$

   Solution: $\{1, 2\} \subseteq \mathbb{N}$

   (c) $\{\mathbb{N}, \mathbb{R}\}$, $\{\mathbb{R}\}$

   Solution: $\{\mathbb{N}, \mathbb{R}\} \supseteq \{\mathbb{R}\}$

   (d) $\{\mathbb{R}\}$, $\{1, 3, 4\}$

   Solution: There is no relation between these two sets.

   (e) $\mathbb{R}$, $\{1, \pi, \sqrt{2}, \sqrt{-1}\}$

   Solution: There is no relation between these two sets ($\sqrt{-1}$ is not a real number).

3. (5 points) For each of the following determine whether or not it is a function, if not explain why not.

(a) $f : A \to B$ where $A = \{1,2,3,4,5\}$ and $B = \{b,x,t,m,z,y,a\}$ given by the following set $\{(1,a),(4,b),(2,b)(5,t),(2,a)\}$.

Solution: No, $2 \mapsto b$ and $2 \mapsto a$.

(b) $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \tan(x)$.

Solution: No. $\tan\left(\frac{\pi}{2}\right)$ has no value in $\mathbb{R}$.

(c) $h : \mathbb{N} \to \mathbb{Z}^{>0}$ given by $h(x) = x - 1$

Solution: No, 0 and 1 do not map to anything.

(d) $k : A \to B$ where $A = \{18, 38, 485, 382385, 25\}$ and $B = \{1,2,3,4,5\}$ given by the following set $\{(18,1),(38,1),(285,1),(382385,1),(25,1)\}$.

Solution: No. $(285,1)$ makes no sense since $285 \notin A$. Additionally $485 \in A$ maps to nothing.

(e) $l : \mathbb{R} \to \mathbb{R}$ given by $l(x) = \log(|x|)$.

Solution: No, $l(0)$ is not defined in the reals.

4. (5 points) Prove that if $X \subseteq Y$ then $X \cap Z \subseteq Y \cap Z$ for all sets X, Y, Z.

Solution: Let X, Y, Z be sets such that $X \subseteq Y$. To show $X \cap Z \subseteq Y \cap Z$ we take an arbitrary element in the first set and show it is also in the second set. So let $r \in X \cap Z$, this tells us that $r \in X$ **AND** $r \in Z$. Since $X \subseteq Y$ we have that $r \in Y$. So $r \in Y$ and $r \in Z$ so that $r \in Y \cap Z$. Thus every element of $X \cap Z$ is in $Y \cap Z$ so $X \cap Z \subseteq Y \cap Z$.

□

5. (5 points) Prove that $\mathcal{P}(X) \cup \mathcal{P}(Y) \subseteq \mathcal{P}(X \cup Y)$. Are they equal? If not give a counterexample.

Solution: Again we choose an arbitrary element of the first set and show it is in the second: Let $r \in \mathcal{P}(X) \cup \mathcal{P}(Y)$. This means that r is a *set*. In particular $r \subseteq X$ or $r \subseteq Y$. In either case $r \subseteq X \cup Y$. Which means that $r \in \mathcal{P}(X \cup Y)$.

□

These sets are NOT always equal, for example choose $X = \{1,2\}$ and $Y = \{a,b,c\}$ then $X \cup Y = \{1,a,2,b,c\}$. So $\{1,a\} \in \mathcal{P}(X \cup Y)$ but $\{1,a\} \notin \mathcal{P}(X) \cup \mathcal{P}(Y)$ since $a \notin X$ and $1 \notin Y$.

6. (20 points) For the following statements either give a proof or a counterexample. The sets X, Y, Z are subsets of a universal set U. Counter examples must also include the definition for U.

(a) For all sets X and Y, either $X \subseteq Y$ or $Y \subseteq X$.

Solution: This is false. Let $U = \{1,2,3\}$ and $X = \{1\}$ and $Y = \{2\}$ then $X \not\subseteq Y$ and $Y \not\subseteq X$.

(b) $\overline{Y\backslash X} = X \cup \overline{Y}$

Solution: This is true. Let U be any universal set and $X, Y \subseteq U$. We show the two given sets are subsets of each other. For reference we expand $\overline{Y\backslash X}$, $a \in \overline{Y\backslash X}$ means that $a \in U$ but $a \notin Y\backslash X$. However a may be in X.

($\subseteq$) If $a \in \overline{Y\backslash X}$ then that is saying that $a \in U\backslash(Y\backslash X)$. That is a is in U and a is not in $Y\backslash X$. If $a \in X$ then we are good, if it is not then that means that $a \in U$, but also we know for a fact that $a \notin Y$, so that $a \in U\backslash Y$ or $a \in \overline{Y}$.

($\supseteq$) If $a \in X \cup \overline{Y}$ we have two cases, either $a \in X$ in which case $a \in \overline{Y\backslash X}$ by above. If $a \in \overline{Y}$ this means that $a \in U\backslash Y$, or $a \in U$ and not in Y. So that $a \in \overline{Y\backslash X}$, a larger set than $\overline{Y}$.

□

(c) $X \cup (Y\backslash Z) = (X \cup Y) \setminus (X \cup Z)$

Solution: This is not true. Consider $U = \{1, 2, 3, 4, 5\}$, $X = \{1, 2, 3\}$, $Y = \{4, 5\}$, $Z = \{3, 5\}$. Then $X \cup (Y\backslash Z) = \{1, 2, 3, 4\}$ but $(X \cup Y) \setminus (X \cup Z) = \{4\}$.

□

(d) $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$

Solution: This is true. Let U be a universal set, and $X, Y, Z \subseteq U$.

($\subseteq$) If $a \in X \times (Y \cup Z)$ then a is of the form (x, b) where $x \in X$ and $b \in Y \cup Z$. Thus $a \in X \times Y$ or $a \in X \times Z$. Thus $a \in (X \times Y) \cup (X \times Z)$

($\supseteq$) If $a \in (X \times Y) \cup (X \times Z)$ then a is of the form (x, b) where $x \in X$ but depending on where a is either $b \in Y$ or $b \in Z$. Thus $a \in X \times (Y \cup Z)$.

□

7. (3 points) For the following problems a function definition is given. You must describe a domain and co-domain that ensures it is actually a function. The co-domain you provide need not be precisely the range.

(a) $m(x) = \log(x)$.

Solution: Define $m : \mathbb{R}^{>0} \to \mathbb{R}$

(b) $n(x) = 12$

Solution: We can define the domain to be any set and the co-domain to be any set containing 12.

(c) $o(x)$ defined by the set $\{(1, 2), (\pi, 3), (12, 3)\}$

Solution: We can use a domain to be any set containing $1, \pi, 12$ and the co-domain to be any set containing $2, 3$.

8. (12 points) For the following functions determine whether they are one-to-one or onto or both or neither.

(a) $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = n + 1$

Solution: This function is injective, if $m \neq n$ then $m+1 \neq n+1$.
This is surjective, given $m \in \mathbb{Z}$ then $f(m-1) = m$.
Thus this is a bijection.

(b) $g : \mathbb{Z} \to \mathbb{Z}$, $g(n) = \lceil \frac{n}{2} \rceil$.

Solution: This function is not one-to-one, $g(0) = 0 = g(1)$.
This function is surjective, given $m \in \mathbb{Z}$ then $g(2m) = \lceil \frac{2m}{2} \rceil = \lceil m \rceil = m$.

(c) $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, $h(m,n) = m - n$.

Solution: This function is not injective, $h(0,0) = 0 = h(2,2)$.
This function is surjective, given $m \in integers$ then $h(m,0) = m$.

(d) $j : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, $j(m,n) = m^2 + n^2 + 2$.

Solution: This function is not injective $j(0,1) = 3 = j(1,0)$.
This function is not surjective there is no $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ to result in a negative integer.

9. (5 points) Give examples of functions from $\mathbb{N}$ to $\mathbb{N}$ that are:

(a) one-to-one, but not surjective

Solution: Choose $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 2n$. This is not surjective because nothing maps to 3. It is injective:
Choose $n \neq m \in \mathbb{N}$ then $2n \neq 2m$.

□

(b) surjective but not injective

Solution: Choose $g : \mathbb{N} \to \mathbb{N}$ defined by:

$$g(n) = \begin{cases} 0 & \text{if } n = 0, 1 \\ n - 1 & \text{otherwise} \end{cases}$$

Then g is clearly not injective, both 0 and 1 map to 0. But is surjective:
Choose $m \in \mathbb{N}$ if $m = 0$ then $f(0) = 0$ otherwise $f(m+1) = m$.

(c) injective and surjective (but not the identity function)

Solution: Choose $h : \mathbb{N} \to \mathbb{N}$ defined by:

$$h(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n = 1 \\ n & \text{otherwise} \end{cases}$$

This is not the identity function since $f(1) = 0 \neq 1$. But it is surjective: choose $m \in \mathbb{N}$ if $m = 0$ then $f(1) = m$, if $m = 1$ then $f(0) = 1$, otherwise $f(m) = m$.

It is also injective, if $n \neq m$ then we can examine by cases, if necessary, to see that f does not map these different values to the same value.

(d) neither injective nor surjective.

Solution: Choose $j : \mathbb{N} \to \mathbb{N}$ defined by $j(n) = 1$. This is not surjective, nothing maps to 2. And this is not injective, everything maps to the same value.

10. (11 points) Prove that the function $f : \mathbb{Z}^{>0} \times \mathbb{Z}^{>0} \to \mathbb{Z}^{>0}$ defined by $f(m, n) = 2^m \cdot 3^n$ is injective but not surjective.

Solution: This is injective: If $(m, n) \neq (a, b)$ then we have cases:

Case 1: $m \neq a$ then $2^m \neq 2^a$. Then no matter how many multiples of three are multiplied on we have that $2^m \cdot 3^n \neq 2^a 3^b$.

Case 2: $n \neq b$ then $3^n \neq 3^b$. Again no matter how many multiples of two are multiplied on we have that $2^m \cdot 3^n \neq 2^a 3^b$.

Thus in any case we have that $f(m, n) \neq f(a, b)$ so we have an injective function.

However this function is not surjective. Only $m = 0$ can make $2^m = 1$ and only $n = 0$ can make $3^n = 1$ but 0 is not a valid value for m or n. Thus this function is not surjective.

Other examples are that no multiplication of multiples of 2 and 3 can result in 5.

□

11. (10 points) Solve the following recurrence relations (provide a closed form solution):

(a) $a_n = -3a_{n-1}$, $a_0 = 4$

Solution: Solving this by reducing terms:

$$\begin{aligned} a_n &= -3a_{n-1} \\ &= -3(-3)a_{n_2} = 9a_{n-2} \\ &= -3(-3)(-3)a_{n-3} = -27a_{n-3} \\ &= \vdots \\ &= (-3)^k a_{n-k} \end{aligned}$$

Letting $k = n$:

$$\begin{aligned} &= (-3)^n a_0 \\ &= (-3)^n 4 \end{aligned}$$

(b) $a_n = a_{n-1} + 1$, $a_0 = 12$

Solution: Again reducing terms:

$$\begin{aligned} a_n &= a_{n-1} + 1 \\ &= a_{n-2} + 1 + 1 \\ &= a_{n-3} + 1 + 1 + 1 \\ &= \vdots \\ &= a_{n-k} + k \end{aligned}$$

Letting $k = n$:

$$\begin{aligned} &= a_0 + n \\ &= 12 + n \end{aligned}$$

12. (10 points) Solve the following recurrence relations (provide a closed form solution):

(a) $a_n = 6a_{n-1} - 8a_{n-2}$, $a_0 = 1$, $a_1 = 0$.

Solution: This problem leads to the characteristic equation:

$$x^2 - 6x + 8 = 0$$

which factors as:

$$(x - 4)(x - 2) = 0$$

and so the solutions are $x = 2, 4$. Thus the general form of our solutions is:

$$a_n = A2^n + B4^n$$

Using our initial conditions we get the two equations:

$$\begin{aligned} 1 &= A + B \\ 0 &= 2A + 4B \end{aligned}$$

we can solve for A using the second equation: $A = -2B$ and plugging into the first equation we get:

$$\begin{aligned} 1 &= -2B + B \\ 1 &= -B \\ B &= -1 \end{aligned}$$

Which gives that $A = 2$. Thus our solution is:

$$a_n = 2(2^n) + (-1)4^n$$

or

$$a_n = 2^{n+1} + (-1)4^n$$

(b) $a_n = 2a_{n-1} + 8a_{n-2}$, $a_0 = 4$, $a_1 = 10$.

Solution: This problem leads to the characteristic equation:

$$x^2 - 2x - 8 = 0$$

which factors as:

$$(x-4)(x+2) = 0$$

and so the solutions are $x = -2, 4$. Thus the general form of our solutions is:

$$a_n = A(-2)^n + B4^n$$

Using our initial conditions we get the two equations:

$$\begin{aligned} 4 &= A + B \\ 10 &= -2A + 4B \end{aligned}$$

If we add two times the first equation to the second equation we get:

$$\begin{aligned} 18 &= 0A + 6B \\ &= 6B \\ B &= 3 \end{aligned}$$

Which gives that $A = 1$. Thus our solution is:

$$a_n = (-2)^n + (3)4^n$$

13. (5 points) Give an example of of two uncountable sets A and B such that $A \cap B$ is:

(a) finite

Solution: Choose $A = [1,2]$ and $B = [2,3]$, since A and B are intervals, subsets of $\mathbb{R}$ then they are uncountable. But $A \cap B = \{2\}$ which is finite.

(b) countably infinite

Solution: Choose $A = \mathbb{R} \times \mathbb{Q}$ and $B = \mathbb{Q} \times \mathbb{R}$ then both A and B are uncountable since they contain $\mathbb{R}$, which is uncountable, but $A \cap B = \mathbb{Q} \times \mathbb{Q}$ which is countable infinite.

(c) uncountably infinite

Solution: Choose $A = [1,3]$ and $B = [2,4]$ then A and B are uncountable, since they are interval subsets of $\mathbb{R}$. $A \cap B = [2,3]$ which is also an interval subset of $\mathbb{R}$ and is thus uncountable.