CSCI 2824 - Discrete Structures
Homework 1

You MUST show your work. If you only present answers you will receive minimal credit. This homework is worth 100pts with 5 bonus points.

Due: Wednesday June 14

1. (4 points) Determine if the following are propositions or not:
   (a) $2 + 5 = 19$

   Solution: Yes.

   (b) $x + 4 > 10$

   Solution: No, this is a predicate. On its own this does not have a truth value, only when something is plugged in for x.

   (c) Audrey Meadows was the original "Alice" in "The Honey-Mooners".

   Solution: Yes.

   (d) This statement is false.

   Solution: No, this is a paradox.

2. (9 points) Find the bitwise OR, AND, and XOR of the following pairs of bit strings.
   (a) 0101 1110, 0010 0001

   Solution:

   - OR—0111 1111
   - AND—0000 0000
   - XOR—0111 1111

   (b) 1111 0000, 1010 1010

   Solution:

   - OR—1111 1010
   - AND—1010 0000
   - XOR—0101 1010

   (c) 0000 0111 0001, 10 0100 1000

   Solution:

   - OR—0010 0111 1001

- AND—0000 0100 0000
- XOR—0010 0011 1001

3. (15 points) Write the truth table for each of the following propositions:

   (a) $\neg(p \wedge q) \vee (r \wedge \neg p)$

Solution:

p	q	r	$\neg(p \wedge q)$	$r \wedge \neg p$	$\neg(p \wedge q) \vee (r \wedge \neg p)$
T	T	T	F	F	F
T	T	F	F	F	F
T	F	T	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	F	T

   (b) $(p \vee q) \wedge \neg p$

Solution:

p	q	$p \vee q$	$\neg p$	$(p \vee q) \wedge \neg p$
T	T	T	F	F
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

   (c) $\neg(p \wedge q) \vee (\neg q \vee r)$

Solution:

p	q	r	$\neg(p \wedge q)$	$\neg q \vee r$	$\neg(p \wedge q) \vee (\neg q \vee r)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

4. Given the following statements, and formulas de-construct the symbolic expressions into words.

p: *Today is Monday*

q: *It is raining*

r: *It is hot*

   (a) (3 points) $\neg p \wedge (q \vee r)$

> **Solution:** Today is not Monday and it is either raining or it is hot (or both).

(b) (7 points) $(p \wedge (q \vee r)) \wedge (r \vee (q \vee p))$

> **Solution:** Today is Monday and it is raining or it is hot (or both) and it is also hot or it is raining or it is Monday.

5. (12 points) There are 3 people, A, B, C. Each person either only tells the truth (is truthful) or only tells lies (is not truthful), independently (that is A being truthful does not imply B is or is not truthful, etc.). They then say the statements:

A: "Exactly one of us is telling the truth."

B: "We are all lying."

C: "The other two are lying."

What can you conclude about the identities about A, B, C? Are they liars or truthful? You should do this problem similar to the one in class with a truth-table considering each possibility and determining if it is viable.

Solution:

A truthful	B truthful	C truthful	A's statement	B's statement	C's statement	viable?
T	T	T	F	F	F	No
T	T	F	F	F	F	No
T	F	T	F	F	F	No
T	F	F	T	F	F	Yes
F	T	T	F	F	F	No
F	T	F	T	F	F	No
F	F	T	T	F	T	No
F	F	F	F	T	T	No

The first 3 are not viable since A lies, while he should be truthful. The fourth is viable, because everyone lives up to expectations. The 5th is not viable since B lies while he should be truthful. The 6th and 7th are not viable because A is truthful when he should lie. The 8th is not viable since B is truthful when he should lie.

6. (5 points (bonus)) This question is trickier, and more of a riddle than a puzzle. Two people, A and B are on an island and they either both tell the truth or both lie. There are also two paths on this island, one leads to certain death, the other to paradise. You are allowed to ask one question to determine your path, which you must then stick with. What question do you ask, that guarantees you go to paradise? Explain why it works.

> **Solution:** Logic is a peculiar case where two wrongs do make a right. The key point here is if A and B are truthful I can trust everything that is said, but if they are liars while I can't trust what they say to a direct question, they will actually correct an indirect answer. For example: I ask the question "If I were to ask your partner which path lead to paradise what would she say?".
>
> For the sake of analysis suppose the *left* path is the path to paradise. If both A and B are truthful, and supposing I asked this of A then he knows B would say the left path (because B is truthful) and so A would tell me the left path (because A is truthful).

If A and B are both liars then A knows that B would say the right path (because B lies) and thus A would say the left path (because A lies).

So in either case after asking my question, whether they are both liars or not they will always tell me the correct path.

7. (6 points) In class we talked about some of De Morgan's Laws, I proved some of them, you should prove the others:

(a) Show that $\neg(p \vee q) \Leftrightarrow (\neg p) \wedge (\neg q)$, you should provide a truth table.

Solution:

p	q	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

Thus since the two columns are the same $\neg(p \vee q)$ is logically equivalent to $\neg p \wedge \neg q$.

(b) Show that $\neg(\exists x\ P(x)) \Leftrightarrow \forall x\ \neg P(x)$, you should provide an argument for this.

Solution: We have four cases to show, in order to show logical equivalence. First: if $\neg(\exists x\ P(x))$ is true then that means that $\exists x\ P(x)$ is false, meaning that there is no value of x which makes $P(x)$ true. This gives that every value of x makes $P(x)$ false, or every value of x makes $\neg P(x)$ true. Thus $\forall x\ \neg P(x)$ is true.

Conversely if $\forall x\ \neg P(x)$ is true then every value of x makes $\neg P(x)$ true, so every value of x makes $P(x)$ false. This means that no value of x makes $P(x)$ true. Which means that $\exists x\ P(x)$ is false, and that $\neg(\exists x\ P(x))$ is then true as well.

By the above paragraphs the two statements are logically equivalent, they have the same truth values.

8. (10 points) Determine whether the following propositions are logically equivalent or not:

(a) $p \to q$ and $\neg q \to \neg p$

Solution:

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Thus since the two columns are the same $p \to q$ is logically equivalent to $\neg q \to \neg p$.

(b) $(p \to q) \wedge (q \to r)$ and $p \to r$

Solution:

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	F	T
T	F	F	F	T	F	F
F	T	T	T	T	T	T
F	T	F	T	f	F	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

Thus since the two columns are different, $(p \to q) \wedge (q \to r)$ is not equivalent to $p \to r$. This may disagree with your intuition however recognize that as an argument it does work. That is whenever $p \to q$ and $q \to r$ are true, so is $p \to r$, it is just that there are some situations when the $\wedge$ is not true, but the implication $p \to r$ is true.

9. (6 points) Write the following propositions symbolically using quantifiers with the predicate $L(x, y)$ to mean x loves y. Do you think either are true?

   (a) Somebody loves everybody.

   Solution: $\exists x \, \forall y \; L(x, y)$. There are probably people who claim this is true for them, but they probably don't *love* everybody.

   (b) Somebody loves somebody.

   Solution: $\exists x \, \exists y \; L(x, y)$. This is certainly true, someone somewhere loves someone else.

10. (6 points) Let $P(x)$ denote x is a professional athlete and $Q(x)$ denote x plays soccer. For each of the following propositions determine their truth values and write the proposition in words.

    (a) $\forall x \; (P(x) \to Q(x))$

    Solution: This proposition says that for every person if they are a professional athlete they are a soccer player. This is false, Alexander Ovechkin is a professional hockey player, who does not play soccer.

    Though some arguments might be able to be made that its impossible to know if they have *ever* played soccer, but that is not the intent of the question.

    (b) $\exists x \; (P(x) \vee Q(x))$

    Solution: This proposition says that there exists a person who is a professional athlete or a soccer player (or both). This is true, as mentioned above Alexander Ovechkin is a professional hockey player.

11. (9 points) Determine the truth value of each statement, where x, y are real numbers.

    (a) $\forall x \, \forall y \; (x^2 < y + 1)$

    Solution: This is a false proposition. As a counterexample consider $x = 5$ and $y = 0$ then we have that $25 < 1$ which is false.

(b) $\forall x \; \exists y \; (x^2 + y^2 = 9)$

Solution: Technically this is not true for real numbers. If x is chosen to be greater than 3 then there is no real y which will put it on the circle of radius 3. However this is true for complex numbers. Let x be an arbitrary complex number. Now we can choose a y (since it is the second quantifier) to be $\sqrt{9 - x^2}$, note that since x was chosen first, y can depend on x. In this case we have that:

$$\begin{aligned} x^2 + y^2 &= x^2 + \left(\sqrt{9 - x^2}\right)^2 \\ &= x^2 + 9 - x^2 \\ &= 9 \end{aligned}$$

(c) $\forall x \; \exists y \; ((x < y) \rightarrow (x^2 < y^2))$

Solution: This is true, and probably has numerous proofs. The easiest is as follows: Let x be an arbitrary real number, now we may choose y second and have it depend on x. Let $y = x - 1$, in this case $x < y$ is false, meaning the proposition $(x < y) \rightarrow (x^2 < y^2)$ is vacuously true.

Other arguments might include statements about choosing a big enough y for a given x, and this is valid, you need to be careful about negative x though.

12. (4 points) Translate the following into symbolic logic using the given variables: "To use the wireless network in the airport you must pay the daily fee unless you are a subscriber to the service."

- w: "You can use the wireless network in the airport"
- d: "You pay the daily fee"
- s: "You are a subscriber to the service"

Solution: This can be written as:

$$(d \vee s) \rightarrow w$$

13. (9 points) In a particular isolated tribal village of 100 people, all of which have blond hair, everyone follows a particular set of rules. Any person that knows their own hair color must leave the village forever. Thus every morning there is a ceremony where the elder asks if anyone knows their hair color, and if they do they are kicked out (and everyone else in the village instantly knows as well).

Every person is completely truthful, and will follow this rule. But since no one wants to be kicked out of the village they go to great lengths to avoid seeing their reflection, and telling others what color their hair is.

Eventually, an archaeologist stumbles upon this isolated village and walks among them for a time. The archaeologist mentions (when the whole village can hear her) "someone village member has blond hair" and then immediately leaves.

What happens (assuming no more discussion on hair color occurs)?

Solution: The way to approach these problems is to simplify the question at first.

Imagine there were only 1 person in the village. Then the villager immediately knows their hair color and will kick themselves out the following morning at the ceremony.

If there are 2 people in the village, each villager knows that they other is blond. Person A expects Person B to leave the following morning and vice versa. When neither leave the following morning Person A immediately knows Person B didn't leave because Person B sees a blond person (Person A) and vice versa. Thus on the second morning both people will leave.

One more simplification, if there are 3 people in our village, none will leave the next morning, since they all see other blond people (worth noting that they also expect no one to leave, since they see 2 blond people). On the second morning no one will leave since they all see 2 blond people (however here they might each expect the other two to leave since they each see two blond people). After no one leaves on the second morning they all know they must all be blond, and on the third morning everyone will leave.

Thus for our actual problem: it will take 100 mornings after the announcement for anyone to leave, and then everyone will leave all together.